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ELEMENTARY TREATISE

ON

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REV. B. BRIDGE, B.D. F.R.S.

FELLOW OF ST. PETER'S COLLEGE, CAMBRIDGE.

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ALGEBRA.

INTRODUCTION.

ALGEBRA is that branch of Mathematical science, in which number and quantity and their several relations are made the subject of calculation, by means of certain signs and symbols, the nature and meaning of which are explained in the following Definitions.

I.

DEFINITIONS.

1. Quantities whose values are *known* or *determined*, are generally expressed by the *first* letters of the Alphabet, a, b, c, d, &c.; and *unknown* or *undetermined* quantities are commonly represented by the *last* letters of the Alphabet, x, y, z, &c.

2. The *multiples* of these quantities, such as, *twice a*, *three times b*, *five times x*, &c. are expressed by placing *numbers* before them thus, $2a, 3b, 5x$, &c.; and the numbers 2,3,5, &c. thus prefixed are called the *coefficients* of a, b, x, &c. in the several quantities $2a$, $3b$, $5x$, &c.

3. The sign + (*plus*) placed between two or more quantities means that those quantities should be *added* together; thus, $a+b+x+$ &c. means the *sum* of the quantities $a, b, x,$ &c.; and the sign − (*minus*) placed before any quantity means that such quantity should be *subtracted* from the quantity or quantities with which it is combined; thus, $a-b$ means the *difference* between a and b; and $a+b-c$, the difference between $a+b$ and c.

4. In the general expression $a+2b-4x+3y-5z$ &c. such quantities as have the sign + prefixed to them are called *positive* or *affirmative* quantities; and such as have the sign − prefixed to them, are called *negative* quantities. If no sign be prefixed to a quantity, then the sign + is understood; thus in the foregoing expression the *positive* quantities are a, $+2b$, $+3y$, and the *negative* ones $-4x$, $-5z$.

5. The general sign for the *multiplication* of quantities is ×; but the manner of expressing the product of two or more quantities is varied according to circumstances. The product of quantities consisting of single letters is expressed by placing those letters one after another, according to the order in which they stand in the Alphabet; thus, the product of a and b is expressed by ab; of a, b, and x, by abx; of $3a$, x, and y, by $3axy$; &c. &c. The product of $a+b$ and $c+d$ is expressed by $\overline{a+b}\times\overline{c+d}$, or $\overline{a+b}\,.\,\overline{c+d}$, or $(a+b)\,(c+d)$; in the two former cases, the line drawn over $a+b$ and $c+d$, to mark them as distinct quantities, is called a *vinculum*.

6. The sign ÷ placed between two quantities means that the former of those quantities is to be *divided* by the latter; thus, $a\div b$ means that a is to be divided by b; $\overline{a+b}\div\overline{c+d}$, that $a+b$ is to be divided by $c+d$. But this division is more simply expressed by making the former quantity the *numerator*,

rator, and the latter the *denominator* of a fraction; thus, $\frac{a}{b}$ expresses the quotient of a and b; and $\frac{a+b}{c+d}$, the quotient of $a+b$ and $c+d$.

7. The *powers* of algebraic quantities are expressed by placing a *small figure* (equivalent to the number of factors, and called the *index* or *exponent* of the power) at the right hand of the letter; thus,

$a \times a$ . . . . or the *square* of a . . is expressed by a^2,
$b \times b \times b$ . . . . . or the *cube* of b . . . . by b^3,
$x \times x \times x \times x$ . . . or the *fourth power* of x . . . . . . by x^4,
$\overline{a+b} \times \overline{a+b} \times \overline{a+b}$ or the *cube* of $a+b$ . . . . by $\overline{a+b}|^3$,
and so on.

8. The *roots* of quantities are expressed by the sign $\sqrt{}$, with the proper index annexed; thus,

$\sqrt[2]{a}$, or $\sqrt{a}$, expresses the *square root* of a,
$\sqrt[3]{b}$ . . . . *cube root* of b,
$\sqrt[4]{a+x}$ . . . . *fourth* or *biquadrate root* of $a+x$,

and so on. The roots of quantities may also be expressed by *fractional indices*; but this method of notation requires an explanation, which will be given in Chap. III.

9. *Like* quantities are such as consist of the *same letter*, or the *same combination of letters*; thus, $5a$ and $7a$; $4ab$ and $9ab$; $2bx^2$ and $6bx^2$; &c. are called *like* quantities; and *unlike* quantities are such as consist of *different letters*, or of *different combinations of letters*; thus, $4a$, $3b$, $7ax$, $5bx^2$, &c. are *unlike* quantities.

10. Algebraic quantities have also different denominations, according to the number of terms (connected by the signs + or −) of which they consist; thus,

a, $2b$, $3ax$, &c. quantities consisting of *one* term, are called *simple* quantities.

$a+x$, a quantity consisting of *two* terms, is called a *binomial.*

$b-c$ (that particular species of binomial which expresses the *difference* between two quantities) is called a *residual.*

$bx+y-z$, a quantity consisting of *three* terms, is called a *trinomial.*

$a^2x+by-3c+d$, a quantity consisting of *four* terms, is called a *quadrinomial.*

$a+b-c+x-y$ &c. a quantity consisting of an indefinite number of terms, a *multinomial.*

11. The sign = placed between two or more quantities, expresses the *equality* of such quantities; thus, "$a+b=c+d$," means that $a+b$ is equal to $c+d$; and "$ax+by=cx+dy=2x+fy$," mean that the quantities $ax+by$, $cx+dy$, and $ex+fy$, are all equal to each other. When quantities are thus connected together by this sign of equality, the expression itself is called an *equation.*

12. In algebraical operations the word *therefore,* or *consequently,* often occurs. To express this word, the symbol $\therefore$ is generally made use of; thus, the sentence "therefore $a+b$ is equal to $c+d$," is expressed by "$\therefore\ a+b=c+d$."

II.

Exemplification of the Algebraic Signs and Symbols.

13. The use of these several *signs, symbols,* and *abbreviations,* may be exemplified in the following manner:

EXAMPLE I.

In the algebraic expression $a+b-c$, let $a=9$, $b=7$, and $c=3$; then

$$\begin{aligned} a+b-c &= 9+7-3 \\ &= 16-3=13. \end{aligned}$$

Ex. 2.

In the expression $ax+ay-xy$, let $a=5$, $x=2$, $y=7$; then, to find its value, we have

$$\begin{aligned} ax+ay-xy &= 5\times2+5\times7-2\times7 \\ &= 10+35-14 \\ &= 45-14=31. \end{aligned}$$

Ex. 3.

What is the value of $\frac{ax+by}{b+x}$, where $a=5$, $b=3$, $x=7$, and $y=5$?

Here $ax+by=5\times7+3\times5=35+15=50$,
and $b+x=3+7=10$;

$$\therefore \frac{ax+by}{b+x}=\frac{50}{10}=5.$$

Ex. 4.

In the expression $\frac{ax^2+b^2}{bx-a^2-c}$, let $a=3$, $b=5$, $c=2$, $x=6$; What is its numerical value?

Here $ax^2+b^2=3\times6\times6+5\times5=108+25=133$,
and $bx-a^2-c=5\times6-3\times3-2=30-9-2=19$;

$$\therefore \frac{ax^2+b^2}{bx-a^2-c}=\frac{133}{19}=7.$$

Ex. 5.

There is a certain algebraic expression consisting of six *terms* connected together by the sign *plus*; the *first* term of it arises from *multiplying* three times the *square* of a by the quantity b; the *second* term is the *sum of the squares* of a and b *divided* by the quantity c; the *third* is the *product* of a, b, and c; the fourth is *two thirds* of the *product* of a and b; the *fifth* arises from *dividing* the *square of* a by the *cube of* b; and the *last* term is a fraction, whose *binomial* numerator is the *difference* between a and b, and whose *trinomial* denominator is the sum of the *cubes* of a and b and the *fourth power* of c.

All this is expressed in one *line* of algebraic writing, thus;

$$3a^2b+\frac{a^2+b^2}{c}+abc+\frac{2ab}{3}+\frac{a^2}{b^3}+\frac{a-b}{a^3+b^3+c^4}.$$

$$\left.\begin{array}{r}\text{Let } a=4,\\ b=3,\\ c=2;\end{array}\right\}\begin{array}{l}\text{then the value of this quantity is,}\\ 144+\frac{16+9}{2}+24+8+\frac{16}{27}+\frac{4-3}{64+27+16},\end{array}$$

or

$$176+\frac{25}{2}+\frac{16}{27}+\frac{1}{107}=189\tfrac{5190}{5778}.$$

CHAP. I.

ON THE ADDITION, SUBTRACTION, MULTIPLICATION AND DIVISION OF ALGEBRAIC QUANTITIES.

---

14. Previous to the application of the fundamental rules of Arithmetic to Algebraic quantities, it may be proper to observe, that the symbols + and − are distinguished from all the other signs or symbols, by giving a kind of *quality* or *affection* to the quantities to which they are annexed. As all those terms which have the sign + prefixed to them are to be *added*, and those which have the sign − prefixed to them are to be *subtracted*, from the terms which precede them, the former have a tendency to *increase*, and the latter to *diminish*, the quantities with which they are combined. A *compound* quantity, $x-a$ for instance, will therefore be positive or negative, according to the effect which it produces upon some third quantity c. Thus, if x be *greater* than a, then $c+x-a$ (since x is *added* and a *subtracted*) is greater than c; if x be *less* than a, then $c+x-a$ is less than c; *i.e.* "if x be *greater* than a, $x-a$ is positive; and if x be *less* than a, then $x-a$ is negative." In the same manner it might be shewn that the expression $a-b+c-d$ is positive or negative, according as $a+c$ is greater or less than $b+d$; and so of all compound quantities whatever.

III.

ADDITION.

From the division of algebraic quantities into *positive* and *negative*, *like* and *unlike*, there arise three cases of Addition.

CASE I.

To add like quantities with like signs.

15. In this case, the rule is "To add the coefficients of "the several quantities together, and to the result annex the "common sign, and the common letter or letters;" for it is evident, from the common principles of Arithmetic, if $+2a$, $+3a$, and $+5a$ be added together, their sum must be $+10a$; and if $-3b^2$, $-4b^2$, and $-8b^2$ be added together, their sum must be $-15b^2$.

Ex. 1.

$$\begin{array}{r}
2x+3a-4b\\
3x+2a-5b\\
4x+8a-7b\\
9x+4a-6b\\
5x+7a-9b\\
\hline
23x+24a-31b\\
\hline\hline
\end{array}$$

Ex. 2.

$$\begin{array}{r}
7x^2+3xy-5bc\\
9x^2+2xy-7bc\\
11x^2+5xy-4bc\\
x^2+4xy-bc\\
x^2+9xy-2bc\\
\hline
29x^2+23xy-19bc\\
\hline\hline
\end{array}$$

Ex. 3.

$$\begin{array}{r}
4a^3-3a^2+1\\
2a^3-a^2+17\\
5a^3-2a^2+4\\
3a^3-7a+3\\
a^3-a^2+10\\
\hline
15a^3-14a^2+35\\
\hline\hline
\end{array}$$

Ex. 4.

$$\begin{array}{r}
3x^3+4x^2-x\\
2x^3+x^2-3x\\
7a^3+2x^2-2x\\
4x^3+x^2-x\\
\hline
\\
\hline\hline
\end{array}$$

Ex. 5.

$$\begin{array}{r}
7a^3-3a^2b+2ab^2-3b^3\\
4a^3-a^2b+ab^2-b^3\\
a^3-2a^2b+3ab^2-5b^3\\
5a^3-3a^2b+4ab^2-2b^3\\
\hline
\\
\hline\hline
\end{array}$$

Ex. 6.

$$\begin{array}{r}
2x^2y-3x+2\\
4x^2y-2x+1\\
3x^2y-5x+10\\
x^2y-x+15\\
\hline
\\
\hline\hline
\end{array}$$

In these Examples it may be observed that some of the quantities have *no coefficient*. In this case, *unity* or 1 *is always understood.* Thus, in adding up the *first* column of Ex. 2. we say, $1+1+11+9+7=29$; in the *third*, $2+1+4+7+5=19$; and so of the rest.

CASE

CASE II.

To add like quantities with unlike signs.

16. Since (by Art. 14) the compound quantity $a+b-c+d-e$ &c. is positive or negative, according as the sum of the positive terms is greater or less than the sum of the negative ones, the aggregate or sum of the quantities $2a-4a+7a-3a$ will be $+2a$, and of the quantities $7b^2-5b^2+2b^2-8b^2$ will be $-4b^2$; for in the former case the excess of the sum of the positive terms above the negative ones is $2a$, and in the latter $4b^2$. Hence this general rule for the addition of like quantities with unlike signs, "Collect "the coefficients of the *positive* terms into one sum, and also "of the *negative*; subtract the *lesser* of these sums from "the *greater*; to this *difference*, annex the sign of the "*greater* together with the common letter or letters, and the "result will be the sum required."

If the aggregate of the positive terms be *equal* to that of the negative ones, then this *difference* is equal to 0; and consequently the sum of the quantities will be equal to 0, as in the *second* column of Ex. 2. following.

Ex. 1.

$$\begin{array}{r} 4x^2-3x+4 \\ -2x^2+x-5 \\ 3x^2-5x+1 \\ 7x^2+2x-4 \\ -x^2-4x+13 \\ \hline 11x^2-9x+9 \\ \hline\hline \end{array}$$

Ex. 2.

$$\begin{array}{r} -7ab+3bc-xy \\ -ab+2bc+4xy \\ 3ab-bc+2xy \\ -2ab+4bc-3xy \\ 5ab-8bc+xy \\ \hline -2ab\quad * \quad+3xy \\ \hline\hline \end{array}$$

Ex. 3.

$$\begin{array}{r} -5x^3+13x^2 \\ -2x^3-4x^2 \\ 7x^3+x^2 \\ 9x^3-14x^2 \\ -13x^3-2x^2 \\ \hline -4x^3-6x^2 \\ \hline\hline \end{array}$$

Ex. 4.

$$\begin{array}{r} 4x^3-2x+3y \\ -x^3+4x-y \\ 7x^3-x+9y \\ 9x^3+21x-2y \\ \hline \\ \hline\hline \end{array}$$

Ex. 5.

$$\begin{array}{r} 5a^3-2ab+b^2 \\ -a^3+ab-2b^2 \\ 4a^3-3ab+b^2 \\ 2a^3+4ab-4b^2 \\ \hline \\ \hline\hline \end{array}$$

Ex. 6.

$$\begin{array}{r} 4x^2y^2+2xy-3 \\ -x^2y^2-xy-1 \\ 3x^2y^2+4xy-5 \\ -9x^2y^2-2xy+9 \\ \hline \\ \hline\hline \end{array}$$

Case III.

17. There now only remains the case where *unlike* quantities are to be added together, which must be done by collecting them together into one line, and annexing their proper signs; thus the sum of $3x$, $-2a$, $+5b$, $-4y$, is $3x-2a+5b-4y$; except when *like* and *unlike* quantities are mixed together, as in the following examples, where the expressions may be simplified, by collecting together such quantities as will coalesce into one sum.

Ex. 1.

$$\left.\begin{array}{l} 3ab+x-y \\ 4c-2y+x \\ 5ab-3c+d \\ 4y+x^2-2y \\ \hline 8ab+2x-y+c+d+x^2 \\ \hline\hline \end{array}\right\}$$

Collecting together *like* quantities, and beginning with $3ab$, we have $3ab+5ab=8ab$; $+x+x=+2x$; $-y-2y+4y-2y=-y$; $4c-3c=+c$; besides which there are the two quantities $+d$ and $+x^2$, which do not coalesce with any of the others; the sum required therefore is $8ab+2x-y+c+d+x^2$.

Ex. 2.

$$\begin{array}{l} 4x^2-2xy+1-3y+4x^3 \\ 4y+3x^3-y^2+xy-x^2 \\ 5x^3-2x+y-15+y^2 \\ \hline 3x^2-xy-14+2y+12x^3-2x \\ \hline\hline \end{array} \left\{\begin{array}{r} \text{Here } 4x^2-x^2=3x^2 \\ -2xy+xy=-xy \\ +1-15=-14 \\ -3y+4y+y=+2y \\ +4x^3+3x^3+5x^3=+12x^3 \\ -y^2+y^2=0 \\ -2x=-2x. \end{array}\right.$$

IV.

SUBTRACTION.

18. If it were required to subtract $5-2$ (i.e. 3) from 9, it is evident that the remainder would be *greater* by 2, than if 5 only were subtracted. For the same reason, if $b-c$

were subtracted from a, the remainder would be greater by c, than if b only were subtracted. Now, if b is subtracted from a, the remainder is $a-b$; and consequently, if $b-c$ be subtracted from a, the remainder will be $a-b+c$. Hence this general Rule for the subtraction of algebraic quantities, “ Change the signs of the quantities *to be subtracted*, and “ then place them one after another, as in Addition.”

Ex. 1.

From $5a+3x-2b$, take $2c-4y$. The quantity to be subtracted *with its signs changed*, is $-2c+4y$; therefore the remainder is $5a+3x-2b-2c+4y$.

Ex. 2.

From $7x^2-2x+5$, take $3x^2+5x-1$.
The remainder is $7x^2-2x+5 \quad -3x^2-5x+1$,
or $7x^2-3x^2-2x-5x+5 \quad +1=4x^2-7x+6$.

But when *like* quantities are to be subtracted from each other, as in Ex. 2., the better way is to set one row under the other, and apply the following Rule; “*Conceive* the “ signs of the *quantities to be subtracted* to be *changed*, and “ then proceed as in Addition.”

	Ex. 3.	Ex. 4.	Ex. 5.
From	$7x^2-2x+5$	$12a^2-3a+b-1$	$5y^2-4y+3a$
Subtract	$3x^2+5x-1$	$6a^2+a-2b+3$	$6y^2-4y-a$
Remainder	$4x^2-7x+6$	$6a^2-4a+3b-4$	$-y^2 \quad * \quad +4a$

	Ex. 6.	Ex. 7.	Ex. 8.
From	$7xy+2x-3y$	$14x+y-z-5$	$13x^3-2x^2+7$
Subtract	$2xy-x+y$	$x+y+z-11$	$-x^3+x^2-6$
Remainder			

MULTI-

V.

MULTIPLICATION.

19. In the multiplication of algebraic quantities, the four following Rules must be observed.

I. When quantities having *like* signs are multiplied together, the sign of the *product* will be $+$; and if their signs are *unlike*, the sign of the *product* will be $-$.*

II. The coefficients of the *factors* must be multiplied together, to form the coefficient of the *product*.

III. The

---

* This rule for the multiplication of the Signs may be thus explained:

I. If $+a$ is to be multiplied by $+b$, it means, that $+a$ is to be *added* to itself as often as there are units in b, and consequently the product will be $+ab$.

II. If $-a$ is to be multiplied by $+b$, it means, that $-a$ is to be *added* to itself as often as there are units in b, and therefore the product is $-ab$.

III. If $+a$ is to be multiplied by $-b$, it means, that $+a$ is to be *subtracted* as often as there are units in b, and consequently the product is $-ab$.

IV. If $-a$ is to be multiplied by $-b$, it means, that $-a$ is to be *subtracted* as often as there are units in b; and, since to *subtract a negative quantity* is the same as *to add a positive one*, the product will be $+ab$.

Or, these Four Rules might be all comprehended in *one*; thus,

To multiply $a-b$ by $c-d$, is to add $a-b$ to itself as often as there are units in $c-d$; now this is done by *adding* it c *times*, and *subtracting* it d *times*;

But $a-b$, *added* c *times* . . . $=ac-bc$,
and $a-b$, *subtracted* d *times* $=-ad+bd$,
$\therefore\ \overline{a-b}\times\overline{c-d}$ . . . . $=ac-bc-ad+bd$.

$$\text{i.e. } +a\times+c=+ac$$
$$-b\times+c=-bc$$
$$+a\times-d=-ad$$
$$-b\times-d=+bd.$$

III. The letters of which they are composed must be set down, one after another, *according to their order in the Alphabet.*

IV. If the *same* letter is found in both factors, the indices of it must be *added* together, to form the index of it in the *product.*

Thus, $+a$ multiplied by $+b$ is equal to $+ab$, and $-a$ multiplied by $-b$ is also equal to $+ab$; $+3x \times -5y = -15xy$; $-3ab \times +4cd = -12abcd$; $-4a^2b^2 \times -3abd^2 = +12a^3b^3d^2$; &c. &c.

From the division of algebraic quantities into *simple* and *compound*, there arise three cases of Multiplication. In performing the operation, the Rule is, "To multiply *first* the signs, *then* the coefficients, and *afterwards* the letters."

CASE I.

20. When *both* factors are *simple* quantities; for which the Rule has been already given.

Ex. 1.	Ex. 2.	Ex. 3.	Ex. 4.
$4ab$	$2axy$	$-3abc$	$-5a^2bc$
$3a$	$-3y$	$5a^2b$	$-2b^2x^2$
$12a^2b$	$-6axy^2$	$-15a^3b^2c$	$+10a^2b^3cx^2$

Ex. 5.	Ex. 6.	Ex. 7.	Ex. 8.
$4abc$	$9x^2y^2$	$-4cdx$	$-7ax^2y$
$3ac$	$-2y$	$-2c$	$-2ac^2x$

CASE II.

21. When one factor is *compound* and the other *simple,* "Then *each term* of the compound factor must be multiplied

"by the simple factor as in the last Case, and the result will "be the product required."

Ex. 1.

$$\begin{array}{ll} \text{Multiply} & 3ab-2ac+d \\ \text{by} & 4a \\ \hline \text{Product} & 12a^2b-8a^2c+4ad \\ \hline\hline \end{array}$$

Ex. 2.

$$\begin{array}{l} 3x^3-2x^2+4 \\ -14ax \\ \hline -42ax^4+28ax^3-56ax \\ \hline\hline \end{array}$$

Ex. 3.

$$\begin{array}{ll} \text{Multiply} & 7x^2-2x+4a \\ \text{by} & -3a \\ \hline \text{Product} & -21ax^2+6ax-12a^2 \\ \hline\hline \end{array}$$

Ex. 4.

$$\begin{array}{l} 12a^3-2a^2+4a-1 \\ 3x \\ \hline \\ \hline\hline \end{array}$$

Ex. 5.

$$\begin{array}{ll} \text{Multiply} & 9a^2x+3a-x+1 \\ \text{by} & -x^2 \\ \hline \text{Product} & \\ \hline\hline \end{array}$$

Ex. 6.

$$\begin{array}{l} 4x^2y+3x-2y \\ -3xy \\ \hline \\ \hline\hline \end{array}$$

CASE III.

22. When *both* factors are *compound* quantities, each term of the multiplicand must be multiplied by each term of the multiplier; and then placing *like quantities under each other,* the sum of all the terms will be the product required.

Ex. 1.

$$\begin{array}{lrr} \text{Multiply} & a+ & b \\ \text{by} & a+ & b \\ \hline \text{1st, by } a \;.\;. & a^2+ & ab \\ \text{2d, by } b \;.\;. & & ab+b^2 \\ \hline \text{Product} & a^2+ & 2ab+b^2 \\ \hline\hline \end{array}$$

Ex. 2.

$$\begin{array}{rrr} a+ & b & \\ a- & b & \\ \hline a^2+ & ab & \\ & -ab & -b^2 \\ \hline a^2 & * & -b^2 \\ \hline\hline \end{array}$$

Ex. 3.

$$\begin{array}{rrrr} a^2+ & ab+ & b^2 & \\ a- & b & & \\ \hline a^3+ & a^2b+ & ab^2 & \\ & -a^2b & -ab^2 & -b^3 \\ \hline a^3 & * & * & -b^3 \\ \hline\hline \end{array}$$

Ex. 4.

$$\begin{array}{l}
3x^2+2x \\
4x+7 \\
\hline
12x^3+8x^2 \\
\qquad +21x^2+14x \\
\hline
12x^3+29x^2+14x \\
\hline\hline
\end{array}$$

Ex. 5.

$$\begin{array}{l}
3x^2-2x+5 \\
6x-7 \\
\hline
18x^3-12x^2+30x \\
\qquad -21x^2+14x-35 \\
\hline
18x^3-33x^2+44x-35 \\
\hline\hline
\end{array}$$

Ex. 6.

$$\begin{array}{l}
14ac-3ab+2 \\
ac-ab+1 \\
\hline
14a^2c^2-3a^2bc+2ac \\
\qquad -14a^2bc \qquad +3a^2b^2-2ab \\
\qquad\qquad +14ac \qquad -3ab+2 \\
\hline
14a^2c^2-17a^2bc+16ac+3a^2b^2-5ab+2 \\
\hline\hline
\end{array}$$

Ex. 7.

$$\begin{array}{l}
x^2-\frac{1}{2}x+\frac{2}{3} \\
\frac{1}{3}x+2 \\
\hline
\frac{1}{3}x^3-\frac{1}{6}x^2+\frac{2}{9}x \\
\qquad +2x^2-x+\frac{4}{3} \\
\hline
\frac{1}{3}x^3+\frac{11}{6}x^2-\frac{7}{9}x+\frac{4}{3} \\
\hline\hline
\end{array}$$

Ex. 8. Multiply $a^3+3a^2b+3ab^2+b^3$ by . . $a+b$.

Answer, $a^4+4a^3b+6a^2b^2+4ab^3+b^4$.

Ex. 9. . . . . . $4x^2y+3xy-1$ . . . by . . $2x^2-x$.

Answ. $8x^4y+2x^3y-2x^2-3x^2y+x$.

Ex. 10. . . . . . x^3-x^2+x-5 . . . . by . . $2x^2+x+1$.

Answ. $2x^5-x^4+2x^3-10x^2-4x-5$.

Ex. 11. Multiply $3a^2+2ab-b^2$ . . by . . $3a^2-2ab+b^2$.
ANSW. $9a^4-4a^2b^2+4ab^3-b^4$.

Ex. 12. . . . . . $x^3+x^2y+xy^2+y^3$ by . . $x-y$.
ANSW. x^4-y^4.

Ex. 13. . . . . . $x^2-\frac{3}{4}x+1$ . . . . by . . $x^2-\frac{1}{2}x$.
ANSW. $x^4-\frac{5}{4}x^3+\frac{11}{8}x^2-\frac{1}{2}x$.

VI.

DIVISION.

23. In the *Division* of algebraic quantities, the same circumstances are to be taken into consideration as in their *multiplication*, and consequently the four following Rules must be observed.

I. That if the signs of the dividend and divisor be *like*, then the sign of the quotient will be $+$; if *unlike*, then the sign of the quotient will be $-$.[*]

II. That the coefficient of the *dividend* is to be divided by the coefficient of the *divisor*, to obtain the coefficient of the *quotient*.

III. That all the letters *common* to both the dividend and the divisor must be *rejected* in the quotient.

IV. That

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[*] The Rule for the *signs* follows immediately from that in Multiplication; thus,

$$\left.\begin{aligned} &\text{If } +a\times+b=+ab, \text{ then } \frac{+ab}{+a}=+b, \text{ and } \frac{+ab}{+b}=+a\\ &\quad +a\times-b=-ab, \ldots \frac{-ab}{+a}=-b, \text{ and } \frac{-ab}{-b}=+a\\ &\quad -a\times-b=+ab, \ldots \frac{+ab}{-a}=-b, \text{ and } \frac{+ab}{-b}=-a \end{aligned}\right\}$$

i.e. *like* signs produce $+$, and *unlike* signs $-$.

IV. That if the same letter be found in both the dividend and divisor with *different* indices, then the index of that letter in the divisor must be *subtracted* from its index in the dividend, to obtain its index in the quotient. Thus,

I. $+abc$ divided by $+ac$ .... or $\frac{+abc}{+ac}=+b$.

II. $+6abc$ .... $-2a$ ... or $\frac{6abc}{-2a}=-3bc$.

III. $-10xyz$ ...... $+5y$ ... or $\frac{-10xyz}{+5y}=-2xz$.

IV. $-20a^2x^2y^3$ .... $-4axy$ .. or $\frac{-20a^2x^2y^3}{-4axy}=+5axy^2$.

Of *Division*, also, there are three Cases; the same as in *Multiplication*.

CASE I.

24. When dividend and divisor are both *simple* terms.

Ex. 1.

Divide $18ax^2$ by $3ax$.

$$\frac{18ax^2}{3ax}=6x.$$

Ex. 2.

Divide $15a^2b^4$ by $-5a$.

$$\frac{+15a^2b^2}{-5a}=-3ab^2.$$

Ex. 3.

Divide $-28x^2y^3$ by $-4xy$.

$$\frac{-28x^2y^3}{-4xy}=+7xy^2.$$

Ex. 4.

Divide $25a^3c^2$ by $-5a^2c$.

$$\frac{+25a^3c^2}{-5a^2c}=$$

Ex. 5.

Divide $-14a^3b^2c$ by $7ac$.

$$\frac{-14a^3b^2c}{+7ac}=$$

Ex. 6.

Divide $-20x^2y^2z^3$ by $-4yz$.

$$\frac{-20x^2y^2z^3}{-4yz.}=$$

CASE II.

25. When the dividend is a *compound* quantity, and the divisor a *simple* one; then each term of the dividend must be divided separately, and the resulting quantities will be the quotient required.

Ex. 1.

Ex. 1.

Divide $42a^2+3ab+12a^2$ by $3a$.

$$\frac{42a^2+3ab+12a^2}{3a}=14a+b+4a.$$

Ex. 2.

Divide $90a^2x^3-18ax^2+4a^2x-2ax$ by $2ax$.

$$\frac{90a^2x^3-18ax^2+4a^2x-2ax}{2ax}=45ax^2-9x+2a-1.$$

Ex. 3.

Divide $4x^3-2x^2+2x$ by $2x$.

$$\frac{4x^3-2x^2+2x}{2x}=$$

Ex. 4.

Divide $-24a^2x^2y-3axy+6x^2y^2$ by $-3xy$.

$$\frac{-24a^2x^2y-3axy+6x^2y^2}{-3xy}=$$

Ex. 5.

Divide $14ab^3+7a^2b^2-21a^2b^3+35a^3b$ by $7ab$.

$$\frac{14ab^3+7a^2b^2-21a^2b^3+35a^3b}{7ab}$$

CASE III.

26. When dividend and divisor are *both compound* quantities. In this case, the Rule is, "to arrange both dividend and "divisor according to the powers of the same letter, beginning "with the *highest;* then find how often the first term of the "divisor is contained in the first term of the dividend, and "place the result in the quotient; multiply each term of the "divisor by this quantity, and place the product under the "corresponding (i: e. *like*) terms in the dividend, and then "subtract it from them; to the remainder bring down as "many terms of the dividend, as will make its number of terms

“ equal to that of the divisor; and then proceed as before, till “ all the terms of the dividend are brought down, as in common “ arithmetic.”

Ex. I.

Divide $a^3-3a^2b+3ab^2-b^3$ by $a-b$.

$$\begin{array}{r|llll}
a-b) & a^3-3a^2b & +3ab^2 & -b^3 & (a^2-2ab+b^2 \\
 & \underline{a^3-\ \ a^2b} & & & \\
 & *\ -2a^2b & +3ab^2 & & \\
 & \underline{\ \ \ -2a^2b} & \underline{+2ab^2} & & \\
 & \ \ \ \ * & \ \ ab^2 & -b^3 & \\
 & & \underline{\ \ ab^2} & \underline{-b^3} & \\
 & & \ \ \ * & \ * & \\
\end{array}$$

In this Example, the dividend is arranged according to the powers of *a*, the first term of the divisor. Having done this, we proceed by the following steps;

I. a is contained in a^3, a^2 times; put this in the quotient.

II. Multiply $a-b$ by a^2, and it gives a^3-a^2b.

III. Subtract a^3-a^2b from a^3-3a^2b, and the remainder is $-2a^2b$.

IV. Bring down the next term $+3ab^2$.

V. a is contained in $-2a^2b$, $-2ab$ times; put this in the quotient.

VI. *Multiply* and *subtract* as before, and the remainder is ab^2.

VII. Bring down the last term $-b^3$.

VIII. a is contained in ab^2, $+b^2$ times; put this in the quotient.

IX. *Multiply* and *subtract* as before, and nothing remains; the *quotient* therefore is $a^2-2ab+b^2$.

Ex. 2.

Ex. 2.

$$a^2+2ax+x^2\,\big)\,a^5+5a^4x+10a^3x^2+10a^2x^3+5ax^4+x^5\,\big(\,a^3+3a^2x+3ax^2+x^3$$
$$a^5+2a^4x+a^3x^2$$
$$*\quad 3a^4x+9a^3x^2+10a^2x^3$$
$$3a^4x+6a^3x^2+3a^2x^3$$
$$*\quad 3a^3x^2+7a^2x^3+5ax^4$$
$$3a^3x^2+6x^2x^3+3ax^4$$
$$*\quad a^2x^3+2ax^4+x^5$$
$$a^2x^3+2ax^4+x^5$$
$$*\quad *\quad *$$

Ex. 3.

$$4x^2-7x\,\big)\,12x^5-13x^4-34x^3+39x^2\,\big(\,3x^3+2x^2-5x+\frac{4x^2}{4x^2-7x}\;^{(a)}$$
$$12x^5-21x^4$$
$$+8x^4-34x^3$$
$$+8x^4-14x^3$$
$$*\quad -20x^3+39x^2$$
$$-20x^3+35x^2$$
$$*\quad +\quad 4x^2$$

Ex. 4.

$$3x+6\,\big)\,6x^4-96\,\big(\,2x^3+4x^2+8x+16$$
$$6x^4-12x^3$$
$$*\quad +12x^3-96$$
$$+12x^3-24x^2$$
$$*\quad +24x^2-96$$
$$+24x^2-48x$$
$$*\quad +48x-96$$
$$+48x-96$$
$$*\quad *$$

(a) When there is a *remainder*, it must be made the *numerator* of a Fraction whose denominator is the *divisor*; this Fraction must then be placed in the *quotient* (with its proper sign), the same as in common Arithmetic.

Ex. 5.

$$\begin{array}{l}
x^2+x-1\big)\,x^6-x^4+x^3-x^2+2x-1\,\big(x^4-x^3+x^2-x+1 \\
\qquad\quad \underline{x^6+x^5-x^4} \\
\qquad\quad * \; -x^5+x^3-x^2 \\
\qquad\qquad\;\; \underline{-x^5-x^4+x^3} \\
\qquad\qquad\;\; * \; +x^4-x^2+2x \\
\qquad\qquad\qquad\;\; \underline{+x^4+x^3-\;x^2} \\
\qquad\qquad\qquad\;\; * \quad -x^3+2x-1 \\
\qquad\qquad\qquad\qquad\;\; \underline{-x^3-\;x^2+x} \\
\qquad\qquad\qquad\qquad\;\; * \quad +\;x^2+x-1 \\
\qquad\qquad\qquad\qquad\qquad\;\; \underline{+\;x^2+x-1} \\
\qquad\qquad\qquad\qquad\qquad\quad\;\; * \quad * \quad *
\end{array}$$

Ex. 6.

$$\begin{array}{l}
x^2-\frac{1}{8}x\big)\,x^4-\frac{5}{4}x^3+\frac{11}{8}x^2-\frac{1}{8}x\,\big(x^2-\frac{3}{4}x+1 \\
\qquad\quad \underline{x^4-\frac{1}{2}x^3} \\
\qquad\quad * \; -\frac{3}{4}x^3+\frac{11}{8}x^2 \\
\qquad\qquad \underline{-\frac{3}{4}x^3+\frac{3}{8}x^2} \\
\qquad\qquad\;\; * \quad +\;x^2-\frac{1}{8}x \\
\qquad\qquad\qquad\;\; \underline{+\;x^2-\frac{1}{2}x} \\
\qquad\qquad\qquad\qquad\; * \quad *
\end{array}$$

Ex. 7. Divide $a^4+4a^3b+6a^2b^2+4ab^3+b^4$ by $a+b$.

ANSWER, $a^3+3a^2b+3ab^2+b^3$.

Ex. 8. . . . . $a^5-5a^4x+10a^3x^2-10a^2x^3+5ax^4-x^5$ by $a^3-3a^2x+3ax^2+x^3$.

ANSW. $a^2-2ax+x^2$.

Ex. 9. . . . $25x^6-x^4-2x^3-8x^2$ by $5x^3-4x^2$.

ANSW. $5x^3+4x^2+3x+2$.

Ex. 10. . . . $a^4+8a^3x+24a^2x^2+32ax^3+16x^4$ by $a+2x$.

ANSW. $a^3+6a^2x+12ax^2+8x^3$.

Ex. 11. Divide a^5-x^5 by $a-x$.

ANSWER, $a^4+a^3x+a^2x^2+ax^3+x^4$.

Ex. 12. . . . . $6x^4+9x^2-20x$ by $3x^2-3x$.

ANSW. $2x^2+2x+5-\frac{5x}{3x^2-3x}$

Ex. 13. . . . . . $9x^6-46x^5+95x^2+150x$ by x^2-4x-5.

ANSW. $9x^4-10x^3+5x^2-30x$.

Ex. 14. . . . . . . $x^4-\frac{13}{6}x^3+x^2+\frac{4}{3}x-2$ by $\frac{4}{3}x-2$.

ANSW. $\frac{3}{4}x^3-\frac{1}{2}x^2+1$.

VII.

On the application of the foregoing Rules to Quantities with literal Coefficients.

27. In applying the foregoing rules to quantities with *literal* coefficients, such as, mx, ny, qx^2 &c. (where m, n, q &c. may be considered as the coefficients of x, y, x^2, &c.) a compound quantity may be expressed by placing the coefficients of *like* quantities one after another (with their proper signs) in a parenthesis, and then annexing the common letter or letters. Thus, the *sum* of mx and nx, which is $mx+nx$, may be expressed by $(m+n)x$; their *difference*, which is $mx-nx$, by $(m-n)x$; multinomial $mx^2+nx^2-px^2+qx^2$, by $(m+n-p+q)x^2$; and the mixed multinomial $pxy+qy^2-rxy+my^2-nxy$, by $(p-r-n)xy+(q+m)y^2$; &c. &c. According to this method of notation, the operations are performed in the following Examples.

Ex. 1.

$$\text{Add}\left\{\begin{array}{rrr} my^2+ & ny+ & z \\ -py^2- & ry+ & nz \\ qy^2+ & my- & vz \\ & +ry- & qz \end{array}\right.$$

$$\underline{\overline{(m-p+q)y^2+(n+m)y+(1+n-v-q)z.}}$$

Ex. 2.

Ex. 2.

$$\begin{array}{rl}
\text{From} & px^3+qx^2-rx+s \\
\text{Subtract} & mx^3-nx^2+tx-v \\ \hline
\text{Remainder} & (p-m)x^3+(q+n)x^2-(r+t)x+s+v.^{(a)} \\ \hline
\end{array}$$

Ex. 3.

$$\begin{array}{rl}
\text{Multiply} & px^2+qx-r \\
\text{by} & mx-n \\ \hline
 & mpx^3+mqx^2-mrx \\
 & \qquad -npx^2-nqx+nr \\ \hline
\text{Product} & mpx^3+(mq-np)x^2-(mr+nq)x+nr. \\ \hline
\end{array}$$

Ex. 4.

$$\begin{array}{rl}
\text{Multiply} & ax^2-bx+c \\
\text{by} & x^2-cx+1 \\ \hline
 & ax^4-bx^3+cx^2 \\
 & \qquad -acx^3+bcx^2-c^2x \\
 & \qquad\qquad +ax^2-bx+c \\ \hline
\text{Product} & ax^4-(b+ac)x^3+(c+bc+a)x^2-(c^2+b)x+c. \\ \hline
\end{array}$$

Ex. 5. (Division.)

$$\begin{array}{r|llll|l}
x^2-cx+1 & ax^4-(b+ac)x^3+ & (c+bc+a(x^2- & (c^2+b)x+c & & ax^2-bx+c \\
 & ax^4-acx^3 & +ax^2 & & & \\ \hline
 & \ast\quad -bx^3+ & (c+bc)x^2- & (c^2+b)x & & \\
 & \quad\;\; -bx^3 & +bcx^2 & -bx & & \\ \hline
 & \qquad \ast & +cx^2 & -c^2x & +c & \\
 & & +cx^2 & -c^2x & +c & \\ \hline
 & & \ast & \ast & \ast & \\ \hline
\end{array}$$

(a) As the sign prefixed to quantities in a parenthesis affects them *all*; when this sign is *negative*, the signs of all those quantities must be changed in putting them into the parenthesis.. Thus, when + tx is subtracted

Ex. 6. Multiply mx^2-nx-r . . . by $nx-r$.

ANSWER, $mnx^3-(n^2+mr)x^2+r^2$.

Ex. 7. Multiply x^3-px^2+qx-r by $x-a$

ANSW. $x^4-(a+p)x^3+(q+ap)x^2-(r-aq)x+ar$.

Ex. 8. Multiply px^2-rx+q . . . by x^2-rx-q.

ANSW. $px^4-(1+p)rx^3+(q+r^2-pq)x^2-q^2$.

Ex. 9. Divide $ax^3-(a^2+b)x^2+b^2$ by $ax-b$.

ANSW. x^2-ax-b.

VIII.

Some general Theorems, deduced by means of the foregoing Rules.

From the clear and distinct manner in which quantity and its several relations are represented throughout every part of an Algebraic operation, the exemplification of its most ordinary rules affords the means of investigating certain general Theorems relating to the *sum*, *difference*, *product*, &c. &c. of numbers, of which the following are examples.

28. Let a and b be any two numbers of which a is the greater and b the lesser, and let their *sum* be represented by s and their *difference* by d,

$$\begin{aligned} \text{Then}\quad & a+b=s \\ \text{and}\quad & a-b=d \end{aligned}$$

$$\therefore \text{by Addition,}\quad \left.\begin{aligned} 2a &= s+d \\ \text{and}\quad a &= \frac{s}{2}+\frac{d}{2} \end{aligned}\right\}$$

by

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tracted from $-rx$, the result is $-rx-tx$; and, as this means that the sum of rx and tx is to be *subtracted*, that *negative* sum is expressed by $-(rx+tx)=-(r+t)x$. For the same reason, any *multinomial* quantity $-mx^2+nx^2-qx^2+rx^2$, when put into a parenthesis with a *negative* sign prefixed, becomes $-(m-n+q-r)x^2$.

$$\left.\begin{aligned}\text{by Subtraction, } 2b &= s-d\\ \text{and } \quad b &= \frac{s}{2}-\frac{d}{2}\end{aligned}\right\}.$$

From which we deduce this general Theorem, that "if the *sum* "and *difference* of any two numbers be given, the *greater* of "them may be found by adding half the given sum to half the "given difference; and the *lesser*, by subtracting half the given "difference from half the given sum."

29. Let a, b, s, d, have the same relation as before, then

$$s=a+b$$
$$d=a-b$$

Hence, by Multiplication, $s\times d=a^2-b^2$ (See Ex. 2. CASE III. p. 14.)

$$\therefore s=\frac{a^2-b^2}{d}$$

$$\text{and } d=\frac{a^2-b^2}{s}.$$

From which it appears, that "if the *sum* and *difference* of "any two numbers be multiplied together, the *product* of that "sum and difference gives the *difference of the squares* of the "two numbers;" and that, "if the difference of the squares "of the two numbers be divided by their *difference*, it gives their "*sum*; and if by their *sum*, it gives their difference."

30. Let the number c be divided into any two parts a and b;

$$\text{Then } c=a+b$$
$$c=a+b$$

$\therefore$ by Multiplication, $c^2=a^2+2ab+b^2$ (See Ex. 1. CASE III. p. 14.)

From which we infer, that "if a number be divided into two "parts, the *square* of the number is equal to the *sum of the* "*squares* of the two parts, together with *twice the product* of "those parts."

31. Let a and b be any two numbers, then,

$$\begin{array}{r} \text{The difference of their } Cubes = a^3 - b^3 \\ \text{their difference} = a - b \\ \hline \end{array}$$

By actual division, $a-b)a^3-b^3\,(a^2+ab+b^2$ (quotient)

$$\begin{array}{r} a^3-a^2b \qquad\qquad \\ \hline +a^2b-b^3 \qquad \\ +a^2b-ab^2 \qquad \\ \hline +ab^2-b^2 \\ +ab^2-b^3 \\ \hline * \quad\; * \\ \hline\hline \end{array}$$

Hence it appears, that "if the *difference of the cubes* of any "two numbers be divided by their *difference*, the *quotient* "arising will be equal to the *sum of the squares* of the two "numbers together with their *product*."

CHAP. II.

ON ALGEBRAIC FRACTIONS.

THE Rules for the management of Algebraic Fractions are the same with those in Common Arithmetic.

IX.

ON THE REDUCTION OF FRACTIONS.

32. *To reduce a mixed Quantity to an improper Fraction.*

RULE. " Multiply the *integral* part by the denominator " of the fraction, and to the *product* annex the numerator with " its proper sign; under this *sum* place the former denominator, " and the result is the improper fraction required."

EXAMPLE I.

Reduce $3a+\frac{2x}{5a^2}$ to an improper fraction.

The integral part × the *denominator* of the fraction + the *numerator* $=3a\times5a^2+2x=15a^3+2x$;

Hence, $\frac{15a^3+2x}{5a^2}$ is the fraction required.

Ex. 2.

Reduce $5x-\frac{4x}{6a^2}$ to an improper fraction.

Here $5x\times6a^2=30a^2x$; to this add the numerator with its proper sign, viz. $-4x$; then $\frac{30a^2x-4x}{6a^2}$ is the fraction required.

Ex. 3.

Reduce $5x - \frac{2x-3}{7}$ to an improper fraction.

Here $5x \times 7 = 35x$. In adding the numerator $2x-3$ *with its proper sign*, it is to be recollected, that the sign $-$ affixed to the fraction $\frac{2x-3}{7}$ means that the *whole* of that fraction is to be *subtracted*, and consequently that the signs of each term of the numerator must be *changed* when it is combined with $35x$; hence the improper fraction required is $\frac{35x-2x+3}{7} = \frac{33x+3}{7}$.

Ex. 4. Reduce $4ab + \frac{2c}{3a}$ to an improper fraction.

Answer, $\frac{12a^2b+2c}{3a}$.

Ex. 5. . . . . $3b^2 - \frac{4a}{5x}$ to an improper fraction.

Answ. $\frac{15b^2x-4a}{5x}$.

Ex. 6. . . . . $a - x + \frac{a^2-ax}{x}$ to an improper fraction.

Answ. $\frac{a^2-x^2}{x}$.

Ex. 7. . . . . $3x^2 - \frac{4x-9}{10}$ to an improper fraction.

Answ. $\frac{30x^2-4x+9}{10}$.

33. *To reduce an improper Fraction to a mixed Quantity.*

Rule. "Observe which terms of the *numerator* are divisible "by the *denominator* without a remainder, the quotient will "give the *integral* part; to this annex (with their proper signs) "the remaining terms of the numerator with the denominator "under them, and the result will be the mixed quantity required."

Ex. 1.

EXAMPLE 1.

Reduce $\frac{a^2+ab+b^2}{a}$ to a mixed quantity.

Here $\frac{a^2+ab}{a}=a+b$ is the *integral* part,

and $\frac{b^2}{a}$ is the *fractional* part;

$\therefore\ a+b+\frac{b^2}{a}$ is the mixed quantity required.

Ex. 2.

Reduce $\frac{15a^2+2x-3c}{5a}$ to a mixed quantity.

Here $\frac{15a^2}{5a}=3a$ is the *integral* part,

and $\frac{2x-3c}{5a}$ is the *fractional* part;

$\therefore\ 3a+\frac{2x-3c}{5a}$ is the mixed quantity required.

Ex. 3. Reduce $\frac{4x^2-5a}{2x}$ to a mixed quantity.

ANSWER, $2x-\frac{5a}{2x}$.

Ex. 4. . . . . $\frac{12a^2+4a-3c}{4a}$ to a mixed quantity.

ANSW. $3a+1-\frac{3c}{4a}$.

Ex. 5. . . . . $\frac{10x^2y+3x^3-2b^2}{x^2}$ to a mixed quantity.

ANSW. $10y+3x-\frac{2b^2}{x^2}$.

34. *To reduce Fractions to a common Denominator.*

RULE. "Multiply each numerator into every denominator "*but its own* for the new numerators, and *all the denominators* "*together* for the common denominator."

Ex. 1.

EXAMPLE I.

Reduce $\frac{2x}{3}$, $\frac{5x}{b}$, and $\frac{4a}{5}$, to a common denominator.

$$\left.\begin{array}{l}2x\times b\times 5=10bx\\ 5x\times 3\times 5=75x\\ 4a\times 3\times b=12ab\end{array}\right\}\text{new numerators;}$$

$$3\ \times b\times 5=15b\ \text{common denominator;}$$

Hence the fractions required are $\frac{10bx}{15b}$, $\frac{75x}{15b}$, $\frac{12ab}{15b}$.

Ex. 2.

Reduce $\frac{2x+1}{5}$, and $\frac{3x}{4}$, to a common denominator.

$$\left.\begin{array}{r}\text{Here }(2x+1)\times 4=\ 8x+4\\ 3x\times 5=15x\end{array}\right\}\text{new numerators;}$$

$$5\times 4=20\ \text{common denominator;}$$

Hence the fractions required are $\frac{8x+4}{20}$, and $\frac{15x}{20}$.

Ex. 3.

Reduce $\frac{5x}{a+x}$, $\frac{a-x}{3}$, and $\frac{1}{2x}$, to a common denominator.

$$\left.\begin{array}{l}\text{Here }5x\times 3\times 2x=30x^2\\ (a-x)\ \ (a+x)\times 2x=2a^2x-2x^3\\ 1\ \times(a+x)\times 3\ =3a\ +3x\\ \hline (a+x)\times 3\ \times 2x\ =6ax+6x^2\end{array}\right\}$$

$\therefore$ the new fractions are $\frac{30x^2}{6ax+6x^2}$, $\frac{2a^2x-2x^3}{6ax+6x^2}$, and $\frac{3a+3x}{6ax+6x^2}$.

Ex. 4. Reduce $\frac{3x}{5}$, $\frac{4bx}{3a}$, and $\frac{5x^2}{a}$, to a common denominator.

ANSW. $\frac{9a^2x}{15a^2}$, $\frac{20abx}{15a^2}$, and $\frac{75ax^2}{15a^2}$.

Ex. 5. Reduce $\frac{2x+3}{x}$, and $\frac{5x+1}{3}$, to a common denominator.

ANSW. $\frac{6x+9}{3x}$, and $\frac{5x^2+x}{3x}$.

Ex. 6.

EXAMPLE I.

Ex. 6. Reduce $\frac{4x^2+2x}{5}$, $\frac{3x^2}{4a}$, and $\frac{2x}{3b}$, to a common denominator.

ANSW. $\frac{48abx^2+24abx}{60ab}$, $\frac{45bx^2}{60ab}$, and $\frac{40ax}{60ab}$.

Ex. 7. Reduce $\frac{7x^2-1}{2x}$, and $\frac{4x^2-x+2}{2a^2}$, to a common denominator.

ANSW. $\frac{14a^2x^2-2a^2}{4a^2x}$, and $\frac{8x^3-2x^2+4x}{4a^2x}$.

35. *To reduce a Fraction to its lowest terms.*

RULE. "Observe what quantity will divide all the terms "both of the numerator and denominator *without a remainder*; "Divide them by this quantity, and the fraction is reduced to "its lowest terms." A more general Rule will be given at the end of this Chapter.

EXAMPLE 1.

Reduce $\frac{14x^3+7ax+21x^2}{35x^2}$ to its lowest terms.

The coefficient of every term of the numerator and denominator of this fraction is divisible by 7, and the letter x also enters into every term; therefore $7x$ will divide both numerator and denominator without a remainder.

Now $\frac{14x^3+7ax+21x^2}{7x}=2x^2+a+3x$,

and $\frac{35x^2}{7x}=5x$;

Hence, the fraction in its lowest terms is $\frac{2x^2+a+3x}{5x}$.

Ex. 2.

Reduce $\frac{20abc-5a^2+10ac}{5a^2c}$ to its lowest term.

Here the quantity which divides both numerator and denominator without a remainder is $5a$; the fraction therefore in its lowest terms is $\frac{4bc-a+2c}{ac}$.

Ex. 3.

Reduce $\frac{a-b}{a^2-b^2}$ to its lowest terms.

Here $a-b$ will divide both numerator and denominator, for by Ex. 2. Case III. page 14. $a^2-b^2=(a+b)(a-b)$; hence $\frac{1}{a+b}$ is the fraction in its lowest terms.

Ex. 4. Reduce $\frac{10x^3}{15x^2}$ to its lowest terms.

Answer, $\frac{2x}{3}$.

Ex. 5. . . . . $\frac{3abx^2}{6ax}$ to its lowest terms.

Answ. $\frac{bx}{2}$.

Ex. 6. . . . . $\frac{14x^2y^2-21x^3y^2}{7x^2y}$ to its lowest terms.

Answ. $\frac{2y-3xy}{x}$.

Ex. 7. . . . . $\frac{51x^3-17x^2+34x}{17x^5}$ to its lowest terms.

Answ. $\frac{3x^2-x+2}{x^4}$.

Ex. 8. . . . . $\frac{a-b}{a^3-b^3}$ to its lowest terms. (See Art. 31).

Answ. $\frac{1}{a^2+ab+b^2}$.

X.

On the ADDITION, SUBTRACTION, MULTIPLICATION, and DIVISION of FRACTIONS.

36. *To add Fractions together.*

Rule. "Reduce the fractions to a common denominator, "and then add their numerators together; bring the re-"sulting fraction to its lowest terms, and it will be the sum "required."

Ex. 1.

EXAMPLE 1.

Add $\frac{3x}{5}$, $\frac{2x}{7}$, and $\frac{x}{3}$, together.

$$\left.\begin{array}{r} 3x\times7\times3=63x \\ 2x\times5\times3=30x \\ x\times5\times7=35x \\ \hline 5\times7\times3=105 \end{array}\right\} \therefore\ \frac{63x+30x+35x}{105}=\frac{128x}{105} \text{ is the fraction required.}$$

Ex. 2. Add $\frac{a}{b}$, $\frac{2a}{3b}$, and $\frac{5b}{4a}$, together.

$$\left.\begin{array}{r} a\times3b\times4a=12a^2b \\ 2a\times b\times4a=\ 8a^2b \\ 5b\times b\times3b=15b^3 \\ \hline b\times3b\times4a=12ab^2 \end{array}\right\} \begin{array}{l} \therefore\ \frac{12a^2b+8a^2b+15b^3}{12ab^2}=\frac{20a^2b+15b^3}{12ab^2} \\ =\text{(dividing by } b\text{) } \frac{20a^2+15b^2}{12ab} \text{ is the} \\ \text{sum required.} \end{array}$$

Ex. 3. Add $\frac{2x+3}{5}$, $\frac{3x-1}{2x}$, and $\frac{4x}{7}$, together.

$$\left.\begin{array}{r} (2x+3)\times2x\times7=28x^2+42x \\ (3x-1)\times5\ \times7=105x-35 \\ 4x\times5\times2x=40x^2 \\ \hline 5\times2x\times7=70x \end{array}\right\} \begin{array}{l} \therefore\ \frac{28x^2+42x+105x-35+40x^2}{70x} \\ =\frac{68x^2+147x-35}{70x} \text{ is the sum} \\ \text{required.} \end{array}$$

Ex 4. Add $\frac{3x}{7}$, $\frac{5x}{9}$, and $\frac{4x}{11}$, together.

ANSWER, $\frac{934x}{693}$.

Ex. 5. . . . $\frac{3a^2}{2b}$, $\frac{2a}{5}$, and $\frac{3b}{7a}$, together.

ANSW. $\frac{105a^3+28a^2b+30b^2}{70ab}$.

Ex. 6. . . . $\frac{2x+1}{3}$, $\frac{4x+2}{5}$, and $\frac{x}{7}$, together.

ANSW. $\frac{169x+77}{105}$.

Ex. 6. Multiply mx^2-nx-r . . . by $nx-r$.

ANSWER, $mnx^3-(n^2+mr)x^2+r^2$.

Ex. 7. Multiply x^3-px^2+qx-r by $x-a$

ANSW. $x^4-(a+p)x^3+(q+ap)x^2-(r-aq)x+ar$.

Ex. 8. Multiply px^2-rx+q . . . by x^2-rx-q.

ANSW. $px^4-(1+p)rx^3+(q+r^2-pq)x^2-q^2$.

Ex. 9. Divide $ax^3-(a^2+b)x^2+b^2$ by $ax-b$.

ANSW. x^2-ax-b.

VIII.

Some general Theorems, deduced by means of the foregoing Rules.

From the clear and distinct manner in which quantity and its several relations are represented throughout every part of an Algebraic operation, the exemplification of its most ordinary rules affords the means of investigating certain general Theorems relating to the *sum, difference, product,* &c. &c. of numbers, of which the following are examples.

28. Let a and b be any two numbers of which a is the greater and b the lesser, and let their *sum* be represented by s and their *difference* by d,

$$\begin{aligned} \text{Then}\quad a+b&=s \\ \text{and}\quad a-b&=d \end{aligned}$$

$$\left.\begin{aligned} \therefore \text{ by Addition, } 2a &= s+d \\ \text{and}\quad a &= \frac{s}{2}+\frac{d}{2} \end{aligned}\right\}.$$

by

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tracted from $-rx$, the result is $-rx-tx$; and, as this means that the sum of rx and tx is to be *subtracted*, that *negative* sum is expressed by $-(rx+tx)=-(r+t)x$. For the same reason, any *multinomial* quantity $-mx^2+nx^2-qx^2+rx^2$, when put into a parenthesis with a *negative* sign prefixed, becomes $-(m-n+q-r)x^2$.

$$\left.\begin{aligned}\text{by Subtraction, } 2b &= s-d\\ \text{and } \quad b &= \frac{s}{2}-\frac{d}{2}\end{aligned}\right\}$$

From which we deduce this general Theorem, that "if the *sum* "and *difference* of any two numbers be given, the *greater* of "them may be found by adding half the given sum to half the "given difference; and the *lesser*, by subtracting half the given "difference from half the given sum."

29. Let a, b, s, d, have the same relation as before, then

$$\begin{aligned}s&=a+b\\ d&=a-b\end{aligned}$$

Hence, by Multiplication, $s\times d=a^2-b^2$ (See Ex. 2. CASE III. p. 14.)

$$\therefore s=\frac{a^2-b^2}{d}$$

$$\text{and } d=\frac{a^2-b^2}{s}.$$

From which it appears, that "if the *sum* and *difference* of "any two numbers be multiplied together, the *product* of that "sum and difference gives the *difference of the squares* of the "two numbers;" and that, "if the difference of the squares "of the two numbers be divided by their *difference*, it gives their "*sum*; and if by their *sum*, it gives their difference."

30. Let the number c be divided into any two parts a and b,

$$\begin{aligned}\text{Then } c&=a+b\\ c&=a+b\end{aligned}$$

$\therefore$ by Multiplication, $c^2=a^2+2ab+b^2$ (See Ex. 1. CASE III. p. 14.)

From which we infer, that "if a number be divided into two "parts, the *square* of the number is equal to the *sum of the* "*squares* of the two parts, together with *twice the product* of "those parts."

Ex. 1.

Multiply $\frac{2x}{7}$ by $\frac{4x}{9}$.

$$\left.\begin{array}{l} 2x\times4x=8x^2 \\ 7\ \times9\ =63 \end{array}\right\} \therefore \text{ the fraction required is } \frac{8x^2}{63}.$$

Ex. 2.

Multiply $\frac{4x+1}{3}$ by $\frac{6x}{7}$.

$$\left.\begin{array}{l} \text{Here} \\ (4x+1)\times6x=24x^2+6x \\ \text{and} \\ 3\times7\ \ =21 \end{array}\right\} \begin{array}{l} \therefore \frac{24x^2+6x}{21}= \text{(dividing the nu-} \\ \text{merator and denominator by 3)} \\ \frac{8x^2+2x}{7} \text{ is the fraction required.} \end{array}$$

Ex. 3.

Multiply $\frac{a^2-b^2}{5b}$ by $\frac{3a^2}{a+b}$.

By Ex. 2. Case III. page 14. $(a^2-b^2)\times3a^2=(a+b)$ $(a-b)\times3a^2$; hence the product is $\frac{3a^2\times(a+b)(a-b)}{5b\times(a+b)}=$ (dividing the numerator and denominator by $a+b$) $\frac{3a^2\times(a-b)}{5b}$ $=\frac{3a^3-3a^2b}{5b}$.

Ex. 4.

Multiply $\frac{3x^2-5x}{14}$ by $\frac{7a}{2x^3-3x}$.

$$\left.\begin{array}{l} \text{Here} \\ (3x^2-5x)\times7a=21ax^2-35ax \\ \text{and} \\ (2x^3-3x)\times14=28x^3-42x \end{array}\right\} \begin{array}{l} \therefore \frac{21ax^2-35ax}{28x^3-42x}= \text{(dividing} \\ \text{the numerator and denomi-} \\ \text{nator by } 7x\text{) } \frac{3ax-5a}{4x^2-6} \text{ is the} \\ \text{fraction required.} \end{array}$$

Ex. 5. Multiply $\frac{2x}{x-1}$ by $\frac{3x}{7}$. Answer, $\frac{6x^2}{7x-7}$.

Ex. 6. Multiply $\frac{3x^2-x}{5}$ by $\frac{10}{2x^2-4x}$. Answer, $\frac{3x-1}{x-2}$.

Ex. 7. . . . . . . $\frac{2a}{a-b}$ by $\frac{a^2-b^2}{8}$. Answ. $\frac{a^2+ab}{4}$.

Ex. 8. . . . . $\frac{3x^2}{5x-10}$ by $\frac{15x-30}{2x}$. Answ. $\frac{9x}{2}$.

39. *On the Division of Fractions.*

Rule. "*Invert* the divisor, and proceed as in Multiplication."

Ex. 1.

Divide $\frac{14x^2}{9}$ by $\frac{2x}{3}$.

Invert the divisor, and it becomes $\frac{3}{2x}$; hence $\frac{14x^2}{9}\times\frac{3}{2x}$ $=\frac{42x^2}{18x}=\frac{7x}{3}$ (dividing the numerator and denominator by $6x$) is the fraction required.

Ex. 2.

Divide $\frac{14x-3}{5}$ by $\frac{10x-4}{25}$.

$$\frac{14x-3}{5}\times\frac{25}{10x-4}=\frac{(14x-3)\times 5}{10x-4}=\frac{70x-15}{10x-4}.$$

Ex. 3.

Divide $\frac{5a^2-5b^2}{2a}$ by $\frac{4a+4b}{6b}$.

$$\frac{5a^2-5b^2}{2a}=\frac{5\times(a+b)\ (a-b)}{2a}$$

$$\frac{4a+4b}{6b}=\frac{4\times(a+b)}{6b};$$

$$\left\{\begin{array}{l}\therefore\ \dfrac{5\times(a+b)\ (a-b)}{2a}\times\dfrac{6b}{4\times(a+b)}\\ =\dfrac{30b\times(a-b)}{8a}=\dfrac{15ab-15b^2}{4a}\ \text{is}\\ \text{the fraction required.}\end{array}\right.$$

Ex. 4. Divide $\frac{4x}{7}$ by $\frac{9x}{5}$. Answer, $\frac{20}{63}$.

Ex. 5. Divide $\frac{4x+2}{3}$ by $\frac{2x+1}{5x}$. Answer, $\frac{10x}{3}$.

Ex. 6. . . . . $\frac{x^2-9}{5}$ by $\frac{x+3}{4}$ . . Answ. $\frac{4x-12}{5}$.

Ex. 7. . . . . $\frac{9x^2-3x}{5}$ by $\frac{x^2}{5}$ . . Answ. $\frac{9x-3}{x}$.

XI.

On the Method of finding the Greatest Common Measure and the Least Common Multiple of two or more Quantities.

40. One quantity is said to *measure* another, when it is contained in that other a certain number of times, without a remainder.

41. A quantity is said to be a *multiple* of another, when it contains that other quantity a certain number of times, without a remainder.

42. The *common measure* of two or more quantities is that quantity which measures them all; and the *greatest* common measure is the greatest quantity which will so measure them. Thus, $2a$ is a common measure of the quantities $24ab^2$, $16a^2bc$, and $12abc^2$, and their *greatest* common measure is $4ab$.

43. A *common multiple* of two or more quantities is that quantity in which each of them is contained without a remainder; and the *least* common multiple is the *least* quantity in which they are so contained. Thus, $40a^3b^2c$ is a common multiple of $5a$, $4ac$, and $2b$, and their *least* common multiple is $20abc$.

44. If one quantity measures another, it will also measure any *multiple* of that quantity. Thus, let b measure a by the units in m, then $a=mb$; and let na be a multiple (denoted by the units in n) of a, then $na=nmb$; consequently b measures na by the units in nm.

45. If one quantity measures two others, it will also measure their sum and difference. For let c measure a by the units in m, and b by the units in n, then $a=mc$, and $b=nc$; therefore $a \pm b$[a]$=mc \pm nc, =(m \pm n)c$; consequently c measures $a+b$ (their *sum*) by the units in $m+n$, and $a-b$ (their *difference*) by the units in $m-n$.

46. Let a and b be any two numbers, whereof a is the greatest; and let the following operation be performed upon them, viz.

$$\left.\begin{array}{llll} b)\,a\,(p & & & \\ \;\;pb & & & \\ \;\;\overline{} & & & \\ & c)\,b\,(q & & \\ & \;\;qc & & \\ & \;\;\overline{} & & \\ & & d)\,c\,(r & \\ & & \;\;rd & \\ & & \;\;\overline{} & \\ & & \;\;0 & \end{array}\right\}$$

Where a divided by b gives the *quotient* p, and *remainder* c; b divided by c, the quotient q, and remainder d; c divided by d, the quotient r, and remainder 0. Then, since in each case the *dividend* is equal to the *divisor* multiplied by the *quotient* plus the *remainder*, we have,

$$c=rd$$

$$b=qc+d= \text{ (for } qc=qrd)\, qrd+d=(qr+1)d$$

$$a=pb+c=\left(\begin{array}{l}\text{for } pb=(pqr+p)d\\ \text{and }\;\; c= \qquad\quad r.d\end{array}\right)(pqr+p+r)d.$$

Hence, since p, q, r are *whole numbers*, d is contained in b as many times as there are units in $qr+1$, and in a as many times as there are units in $pqr+p+r$; consequently the *last divisor* d is a common measure of a and b; and this is evidently the case, whatever be the length of the operation, provided that it be carried on till the remainder is nothing.

This last divisor d is also the *greatest* common measure of a and b. For let x be a common measure of a and b, such that $a=mx$, and $b=nx$, then

$$c=a-pb=mx-pnx=(m-pn)x$$

$$d=b-qc=nx-(qm-pqn)x=(n-qm+pqn)x;$$ $\therefore x$ measures

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[a] The quantity $a \pm b$ means $a+$ or $-b$.

sures d by the units in $n-qm+pqn$, and as it also measures a and b, the numbers a, b and d have a common measure. Now the *greatest* common measure of d is *itself*; consequently d is the greatest common measure of a and b. Hence this Rule for finding the greatest common measure of two numbers. "Divide the greater by the lesser, and the preceding divisor by "the last remainder, till nothing remains; the last divisor is "the greatest common measure."

To find the greatest common measure of *three* numbers, a, b, c; let d be the greatest common measure of a and b, and x the greatest common measure of d and c; then x is the greatest common measure of a, b, and c. For, as a, b, and d, have a common measure; if d and c have also a common measure, that same number will measure a, b, and c; and if x be the *greatest* common measure of d and c, it will also be the greatest common measure of a, b, and c.

In general, let there be any set of numbers, a, b, c, d, e, &c.; and let x be the greatest common measure of a and b; y the greatest common measure of x and c; z the greatest common measure of y and d; &c. &c.; then will y be the greatest common measure of a, b, c; z the greatest common measure of a, b, c, d; &c. &c.

47. To find the greatest common measure of *Algebraic* quantities, the Rule, with respect to *simple* quantities, is, "to "find the greatest common measure of their coefficients, and "then annex to it the letters common to all the quantities;" thus the greatest common measure of $24ax^2y^2$, $16bxy$, and $6axy^2$, is $2xy$.

The operation for finding the greatest common measure of *compound* algebraic quantities is the same as that for finding the greatest common measure of two *numbers*, except that "the remainders which arise in the operation are to be divided "by their simple divisors." In commencing the operation, the terms of which they are composed must be arranged according to the dimensions of the same letter, as in the following examples.

Ex. I.

Find the greatest common measure of $6a^2+11ax+3x^2$ and $6a^2+7ax-3x^2$.

$$\begin{array}{r|l}
6a^2+7ax-3x^2 & 6a^2+11ax+3x^2\,(1 \\
 & \underline{6a^2+\;\;7ax-3x^2} \\
 & \qquad +\;\;4ax+6x^2
\end{array}$$

([a]) Dividing $4ax+6x^2$ by the simple divisor $2x$, we proceed thus;

$$\begin{array}{r|l}
2a+3x & 6a^2+7ax-3x^2\,(3a-x \\
 & \underline{6a^2+9ax} \\
 & \quad -2ax-3x^2 \\
 & \quad \underline{-2ax-3x^2} \\
 & \qquad *\quad *
\end{array}$$

Hence $2a+3x$ is the greatest common measure; and the fraction $\frac{6a^2+\;7ax-3x^2}{6a^2+11ax+3x^2}$, reduced to its lowest terms, is $\frac{3a-x}{3a+x}$.

Ex.

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[a] The reason for rejecting these simple divisors in the course of the operation, is this; It appears by Art. 45, that whatever quantity measures $6a^2+11ax+3x^2$ and $6a^2+7ax-3x^2$ will also measure their *difference* $4ax+6x^2$. Whenever, therefore, a remainder occurs which will not measure both the dividend and divisor, as is evidently the case in the present instance with $4ax+6x^2$ (for $4ax$ cannot possibly divide $6a^2$), it must be reduced to a more simple form by dividing its terms by the quantity which is common to them all. If, after such reduction, it will not measure both the dividend and divisor, it shews that the quantities proposed have no common measure.

Ex. 2.

Find the greatest common measure of $3x^3-24x-9$, and $2x^3-16x-6$.

$$\begin{array}{rll}
2x^3-16x-6\big) & 3x^3-24x-9\,\big(1 & \\
 & \underline{2x^3-16x-6} & \\
 & x^3-\ 8x-3\big) & 2x^3-16x-6\,\big(2 \\
 & & \underline{2x^3-16x-6} \\
 & & \underline{\underline{*\quad\ *\quad\ *}}
\end{array}$$

The greatest common measure, therefore, is x^3-8x-3, and the fraction $\frac{2x^3-16x-6}{3x^3-24x-9}$ in its lowest terms is $\frac{2}{3}$.

If it were required to find the greatest common measure of *more than two* compound algebraic quantities, the rule of course would be the same as that for finding the greatest common measure of the numbers a, b, c, d, e &c. in Art. 46.

48. "For the purpose of finding the least common multiple of two or more numbers;" let a and b be any two numbers whose greatest common measure is x, and let $a=mx$, $b=nx$; then mnx is a multiple of a by the units in n, and of b by the units in m; consequently it is a *common multiple* of a and b. But since x is the greatest common measure of a and b, m and n can have no common divisor; mnx is, therefore, the *least* common multiple of a and b. Now $mx=a$, $nx=b$; $\therefore$ $mnx^2=ab$, and $mnx=\frac{ab}{x}$; hence "the least common "multiple of two numbers is equal to their *product* divided by "their *greatest common measure.*"

To find the least common multiple of 3 numbers a, b, c; let x be the greatest common measure of a and b, and find the greatest common measure of $\frac{ab}{x}$ (the least common multiple of a and b) and c, which call y; then by the Rule $\frac{ab}{x}\times\frac{c}{y}$ or $\frac{abc}{xy}$ is the least common multiple of $\frac{ab}{x}$ and c, and consequently of a, b, c.

Or in general, let a, b, c, d, e &c. be any set of numbers; and let x be the *greatest common measure* of a and b.

y . . . . of $\frac{ab}{x}$ and c,

z . . . . of $\frac{abc}{xy}$ and d,

&c. &c.

then will $\frac{ab}{x}$ be the *least common multiple* of a and b.

$\frac{abc}{xy}$ . . . . of a, b and c.

$\frac{abcd}{xyz}$ . . . . of a, b, c, d.

&c. &c.

CHAP. III.

ON INVOLUTION AND EVOLUTION.

XII.

On the Involution of Numbers and Simple Algebraic Quantities.

49. *Involution,* or "the raising of a quantity to a given power," is performed by the continued multiplication of that quantity into itself, till the number of factors amounts to the number of units in the index of that given power. Thus, the *square* of $a=a^2=a\times a$; the *cube* of $b=b^3=b\times b\times b$; the *fourth power* of $2=2\times 2\times 2\times 2=16$; the *fifth power* of $3=3\times 3\times 3\times 3\times 3=243$; &c. &c. This rule as applied to *numbers* will be readily understood by the mere inspection of the following Table.

ROOTS AND POWERS OF NUMBERS.

Roots	1	2	3	4	5	6	7	8	9	10
Square	1	4	9	16	25	36	49	64	81	100
Cube	1	8	27	64	125	216	343	512	729	1000
4th power	1	16	81	256	625	1296	2401	4096	6561	10000
5th power	1	32	243	1024	3125	7776	16807	32768	59049	100000

50. The operation is performed in the same manner for simple algebraic quantities, except that in this case it must be observed, that the powers of *negative* quantities are alternately +

+ and − ; the *even* powers being positive, and the *odd* powers *negative*. Thus the *square* of $+2a$ is $+2a\times+2a$ or $+4a^2$; the square of $-2a$ is $-2a\times-2a$ or $+4a^2$; but the *cube* of $-2a=-2a\times-2a\times-2a=+4a\times-2a=-8a^3$.

The several powers of $\frac{a}{b}$ are,

$$Square=\frac{a}{b}\times\frac{a}{b}=\frac{a^2}{b^2},$$

$$Cube=\frac{a}{b}\times\frac{a}{b}\times\frac{a}{b}=\frac{a^3}{b^3},$$

$$4th\ power=\frac{a}{b}\times\frac{a}{b}\times\frac{a}{b}\times\frac{a}{b}=\frac{a^4}{b^4},$$

&c. =&c.

And the several powers of $-\frac{b}{2c}$,

$$Square=-\frac{b}{2c}\times-\frac{b}{2c}=+\frac{b^2}{4c^2},$$

$$Cube=-\frac{b}{2c}\times-\frac{b}{2c}\times-\frac{b}{2c}=-\frac{b^3}{8c^3},$$

$$4th\ power=-\frac{b}{2c}\times-\frac{b}{2c}\times-\frac{b}{2c}\times-\frac{b}{2c}=+\frac{b^4}{16c^4}.$$

Upon this principle, the powers of the several roots in the following Table are calculated.

ROOTS AND POWERS OF SIMPLE ALGEBRAIC QUANTITIES.

Roots	a	$-b^2$	$2b^2$	$\frac{a}{2b}$	$-\frac{3x^2}{y}$	$\frac{2a}{3b}$	a^2b	$-\frac{a^2}{b^3}$	$-\frac{3x}{5}$	$\frac{x}{4y}$
Square	a^2	$+b^2$	$4b^4$	$\frac{a^2}{4b^2}$	$+\frac{9x^4}{y^2}$	$\frac{4a^2}{9b^2}$	a^4b^2	$+\frac{a^4}{b^6}$	$+\frac{9x^2}{25}$	$\frac{x^2}{16y^2}$
Cube	a^3	$-b^3$	$8b^6$	$\frac{a^3}{8b^3}$	$-\frac{27x^6}{y^3}$	$\frac{8a^3}{27b^3}$	a^6b^3	$-\frac{a^6}{b^9}$	$-\frac{27x^3}{125}$	$\frac{x^3}{64y^3}$
4th Power	a^4	$+b^4$	$16b^8$	$\frac{a^4}{16b^4}$	$+\frac{81x^8}{y^4}$	$\frac{16a^4}{81b^4}$	a^8b^4	$+\frac{a^8}{b^{12}}$	$+\frac{81x^4}{625}$	$\frac{x^4}{256y^4}$
5th Power	a^5	$-b^5$	$32b^{10}$	$\frac{a^5}{32b^5}$	$-\frac{243x^{10}}{y^5}$	$\frac{32a^5}{243b^5}$	$a^{10}b^5$	$-\frac{a^{10}}{b^{15}}$	$-\frac{243x^5}{3125}$	$\frac{x^5}{1024y^5}$

XIII.

On the Involution of Compound Algebraic Quantities.

51. The powers of compound algebraic quantities are raised

by the mere application of the Rule for Compound Multiplication (Art. 22.) Thus,

Ex. 1. What is the square of $a+2b$?

$$\begin{array}{rl}
 & a+2b \\
 & a+2b \\ \cline{2-2}
 & a^2+2ab \\
 & \quad\;\, +2ab+4b^2 \\ \cline{2-2}
\text{Square}= & a^2+4ab+4b^2 \\ \cline{2-2}
\end{array}$$

Ex. 2. What is the cube of a^2-x?

$$\begin{array}{rl}
 & a^2-x \\
 & a^2-x \\ \cline{2-2}
 & a^4-a^2x \\
 & \quad\;\; -a^2x+x^2 \\ \cline{2-2}
\text{Square}= & a^4-2a^2x+x^2 \\
 & a^2-x \\ \cline{2-2}
 & a^6-2a^4x+a^2x^2 \\
 & \quad\;\; -a^4x+2a^2x^2-x^3 \\ \cline{2-2}
\text{Cube}= & a^6-3a^4x+3a^2x^2-x^3 \\ \cline{2-2}
\end{array}$$

Ex. 3.

What is the 5th power of $a+b$?

$$\begin{array}{llllll l}
a & +b & & & & & \\
a & +b & & & & & \\ \cline{1-2}
a^2 & +ab & & & & & \\
 & +ab & +b^2 & & & & \\ \cline{1-3}
a^2 & +2ab & +b^2 & & & & =\text{Square} \\
a & +b & & & & & \\ \cline{1-3}
a^3 & +2a^2b & +ab^2 & & & & \\
 & +a^2b & +2ab^2 & +b^3 & & & \\ \cline{1-4}
a^3 & +3a^2b & +3ab^2 & +b^3 & & & =\text{Cube} \\
a & +b & & & & & \\ \cline{1-4}
a^4 & +3a^3b & +3a^2b^2 & +ab^3 & & & \\
 & +a^3b & +3a^2b^2 & +3ab^3 & +b^4 & & \\ \cline{1-5}
a^4 & +4a^3b & +6a^2b^2 & +4ab^3 & +b^4 & & =4^{\text{th}}\text{ Power} \\
a & +b & & & & & \\ \cline{1-5}
a^5 & +4a^4b & +6a^3b^2 & +4a^2b^3 & +ab^4 & & \\
 & +a^4b & +4a^3b^2 & +6a^2b^3 & +4ab^4 & +b^5 & \\ \cline{1-6}
a^5 & +5a^4b & +10a^3b^2 & +10a^2b^3 & +5ab^4 & +b^5 & =5^{\text{th}}\text{ Power.} \\ \hline
\end{array}$$

Ex. 4. The 4th *power* of $a+3b$ is $a^4+12a^3b+54a^2b^2+108ab^3+81b^4$.

Ex. 5. The *square* of $3x^2+2x+5$ is $9x^4+12x^3+34x^2+20x+25$.

Ex. 6. The *cube* of $3x-5$ is $27x^3-135x^2+225x-125$.

Ex. 7. The *cube* of x^2-2x+1 is $x^6-6x^5+15x^4-20x^3+15x^2-6x+1$.

52. In the involution of a binomial quantity of the form $a+b$, the several terms in each successive power are found to bear a certain relation to each other, and to observe a certain law, which the following Table is intended to explain.

TABLE OF THE POWERS OF $a+b$.

Powers.	Mode of expressing them.	Powers expanded.
Square	$(a+b)^2$	$a^2+2ab+b^2$.
Cube	$(a+b)^3$	$a^3+3a^2b+3ab^2+b^3$.
4th Power	$(a+b)^4$	$a^4+4a^3b+6a^2b^2+4ab^3+b^4$.
5th Power	$(a+b)^5$	$a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5$.
6th Power	$(a+b)^6$	$a^6+6a^5b+15a^4b^2+20a^3b^3+15a^2b^4+6ab^5+b^6$.

The successive powers of $a-b$ are precisely the same as those of $a+b$, except that the signs of the terms will be alternately $+$ and $-$. Thus, the 4th power of $a-b$ is $a^4-4a^3b+6a^2b^2-4ab^3+b^4$; and so of the rest.

In reviewing that column of the foregoing Table which contains the powers of $a+b$ *expanded*, we may observe,

I. That in each case, the *first* term is a raised to the *given power*, and the *last* term is b raised to the *same* power; thus, in the *square*, the *first* term is a^2, and the *last* b^2; in the cube, the *first* term is a^3, and the *last* b^3; and so of the rest.

II. That, with respect to the intermediate terms, the powers of a *decrease*, and the powers of b *increase*, by unity, in each successive term. Thus, in the fifth power, we have

In the *second* term . . . . $a^4 b$;
third . . . . $a^3 b^2$;
fourth . . . . $a^2 b^3$;
fifth . . . . $a b^4$;

and so in the *other* powers.

III. That in each case, the *coefficient of the second term* is the same with the *index of the given power*. Thus, in the square it is 2; in the cube it is 3; in the fourth power it is 4; and so of the rest.

IV. That if the *coefficient* of a in any term be multiplied by its *index*, and the product divided by the *number of terms to that place*, the quotient will give the *coefficient* of the *next term*. Thus,

In the *fourth* power, $\dfrac{\text{coeff. of } a \text{ in the 2}^{\text{d}}\ \textit{term} \times \textit{its index}}{\text{number of terms to that place}}$

$$=\frac{4\times 3}{2}=\frac{12}{2}=6=\text{coefficient of } \textit{third term}.$$

In the *sixth* power, $\dfrac{\text{coeff. of } a \text{ in the 4}^{\text{th}}\ \textit{term} \times \textit{its index}}{\text{number of terms to that place}}$

$$=\frac{20\times 3}{4}=\frac{60}{4}=15=\text{coefficient of } \textit{fifth term}.$$

We are thus furnished with a general Rule for raising the binomial $a+b$ to *any* power, without the process of actual multiplication. For instance, let it be required to raise $a+b$ to the *eighth* power; then, according to the Rule just laid down,

The *first* term is . . . . a^8.
The *second* . . . . $8a^7b$.
The *third* . . . . $\frac{8\times7}{2}a^6b^2=28a^6b^2$.
The *fourth* . . . . $\frac{28\times6}{3}a^5b^3=56a^5+b^3$.
The *fifth* . . . . $\frac{56\times5}{4}\quad=70a^4b^4$;

and so on.

And thus we have

$$(a+b)^8=a^8+8a^7b+28a^6b^2+56a^5b^3+70a^4b^4+56a^3b^5+28a^2b^6+8ab^7+b^8.$$

In the same manner it will be found,

Ex. 2. That $(a-b)^7=a^7-7a^6b+21a^5b^2-35a^4b^3+35a^3b^4-21a^2b^5+7ab^6-b^7$.

Ex. 3. That $(x-y)^9=x^9-9x^8y+36x^7y^2-84x^6y^3+126x^5y^4-126x^4y^5+84x^3y^6-36x^2y^7+9xy^8-y^9$.

Ex. 4. That $(x+a)^{10}=x^{10}+10x^9a+45x^8a^2+120x^7a^3+210x^6a^4+252x^5a^5+210x^4a^6+120x^5a^7+45x^2a^8+10xa^9+a^{10}$.

In reviewing these several Examples, it may be observed, that, when the number of terms in the resulting quantity is *even*, the * coefficients of the two middle terms is the *same;* and that *in all cases* the coefficients *increase* as far as the *middle* term, and then *decrease* precisely in the same manner until we come to the last term. By attending to this *law of the coefficients*, it will only be necessary to calculate them as far as the *middle term*, and then set down the rest in an *inverted order*. Thus, in Ex. 2. $(\overline{x-y}|^9)$,

The *first* five coefficients are 1, 9, 36, 84, 126.
The *last* five . . . . 126, 84, 36, 9, 1.

"In the even powers, the greatest coefficient is exactly in the middle; but in the odd powers, two coefficients, equal & greater than the others, are found in the middle, belonging to the mean terms." See Euler. Page 100. Wm. Warland.

53. But we are not yet arrived at the *most general* form in which this Rule may be exhibited. Suppose it was required to raise the binomial $a+b$ to any power denoted by the number (n). Proceeding with n as we have done with the several indices in the preceding examples, it appears that,

The *first* term would be a^n.

The *second* . . . . $na^{n-1}b$.

The *third* . . . . . . $\frac{n(n-1)}{2}a^{n-2}b^2$.

The *fourth* . . . . . . $\frac{n(n-1)(n-2)}{2.3}a^{n-3}b^3$.

The *fifth* . . . . $\frac{n(n-1)(n-2)(n-3)}{2.3.4}a^{n-4}b^4$.

The *last* . . . . b^n.

Or, $(a+b)^n=a^n+na^{n-1}b+\frac{n(n-1)}{2}a^{n-2}b^2+\frac{n(n-1)(n-2)}{2.3}a^{n-3}b^3+\frac{n(n-1)(n-2)(n-3)}{2.3.4}a^{n-4}b^4+\&c. \dots +b^n$.

By the same process, $(a-b)^n=a^n-na^{n-1}b+\frac{n(n-1)}{2}a^{n-2}b^2-\frac{n(n-1)(n-2)}{2.3}a^{n-3}b^3+\&c.$; the signs of the terms being alternately + and −.

This general and compendious method of raising a binomial quantity to any given power, is called, from the name of its celebrated inventor, Sir I. NEWTON's "Binomial Theorem[a]." Its use will appear from the following Examples.

EXAMPLE 1.

Raise x^2+3y^2 to the *fifth* power.

In comparing $(x^2+3y^2)^5$ with $(a+b)^n$, we have, $a=x^2$, $b=3y^2$, $n=5$.

---

[a] The demonstration of this Theorem, with its application to the finding the powers and roots of compound quantities in general, forms the subject of Chap. IX.

Substituting these quantities for a, b, n in the foregoing general formula, it appears, that

The *first* term $\Big\}$ . . (a^n) . . . . is $(x^2)^5$ . . $=x^{10}$.

2*d* . . . . . $(na^{n-1}b)$ . . . . is $5\times(x^2)^4\times 3y^2=15x^8y^2$.

3*d* . . . . . $\left(\frac{n(n-1)}{2}a^{n-2}b^2\right)$ . . . . . is $5\times\frac{4}{2}\times(x^2)^3\times(3y^2)^2=90x^6y^4$.

4*th* . . . . $\left(\frac{n(n-1)(n-2)}{2.3}a^{n-3}b^3\right)$ is $5\times\frac{4}{2}\times\frac{3}{3}\times(x^2)^2\times(3y^2)^3=270x^4y^6$.

5*th*, $\left(\frac{n(n-1)(n-2)(n-3)}{2.3.4}a^{n-4}b^4\right)$ is $5\times\frac{4}{2}\times\frac{3}{3}\times\frac{2}{4}\times x^2\times(3y^2)^4=405x^2y^8$.

Last . . . (b^n) . . . . is $(3y^2)^5=243y^{10}$.

So that $(x^2+3y^2)^5=x^{10}+15x^8y^2+90x^6y^4+270x^4y^6+405x^2y^8+243y^{10}$.

In the application of this formula, it may be observed, that the *number of terms* of which the binomial consists, is always *one more* than the *index of the given power;* after having calculated therefore as many terms as there are units in the index of the given power, we may immediately proceed to the *last term.*

Ex. 2.

Raise $3x+2y$ to the 6^{th} power.

Here $\left.\begin{array}{r}3x=a\\ 2y=b\\ n=6\end{array}\right\}$ and $(3x+2y)^6=729x+2916x^5y+4860x^4y^2$ $+4320x^3y^3+2160x^2y^4+576xy^5+64y^6$.

Ex. 3.

Raise $x-2y^2$ to the 7^{th} power.

Here $\left.\begin{array}{r}x=a\\ 2y^2=b\\ n=7\end{array}\right\}$ and comparing $(x-2y^2)^7$ with $(a-b)^n$, we have $x^7-14x^6y^2+84x^5y^4-280x^4y^6+560x^3y^8-$
12 $672x^2y^{10}+448xy^{12}-128y^{14}$ for the quantity required.

54. By means of this Theorem, we are enabled to raise a *trinomial* or *quadrinomial* quantity to any power, without the process of actual multiplication. Thus, suppose it was required

to *square* $a+b+c$; inclosing it in a parenthesis $(a+b)$, and considering it as *one* quantity, we should have $(a+b+c)^2 = \overline{(a+b)+c}|^2 = (a+b)^2+2(a+b)c+c^2 = a^2+2ab+b^2+2ac+2bc+c^2$.

In the same manner we have,

Ex. 1. $(a+b+c+d)^2 = \overline{(a+b)+(c+d)}|^2 = (a+b)^2+2(a+b)(c+d)+(c+d)^2 = a^2+2ab+b^2+2ac+2ad+2bc+2bd+c^2+2cd+d^2$.

Ex. 2. $(a+b+c)^3 = \overline{(a+b)+c}|^3 = (a+b)^3+3(a+b)^2c+3(a+b)c^2+c^3 = a^3+3a^2b+3ab^2+b^3+3a^2c+6abc+3b^2c+3ac^2+3bc^2+c^3$.

Ex. 3. $(x+y+3z)^2 = \overline{(x+y)+3z}|^2 = (x+y)^2+2(x+y)\times 3z+(3z)^2 = x^2+2xy+y^2+6xz+6yz+9z^2$.

XIV.

On the Evolution of Algebraic Quantities.

55. *Evolution,* "or the Rule for extracting the root of any quantity," is just the reverse of *Involution;* and to perform the operation, we must inquire what quantity multiplied into itself, till the number of factors amount to the number of units in the index of the given root, will generate the quantity whose root is to be extracted.

56. This rule, as applied to small numbers and *simple* algebraic quantities, may be easily explained by reference to the Tables in pages 44, 45. Thus,

$49=7\times7$; ∴ the *square root* of 49 (or by Defⁿ. 8, $\sqrt{49}$) $=7$.

$-b^3=-b\times-b\times-b$; ∴ the *cube root* of $-b^3$ ($\sqrt[3]{-b^3}$) $=-b$,

$\frac{16a^4}{81b^4}=\frac{2a}{3b}\times\frac{2a}{3b}\times\frac{2a}{3b}=\frac{2a}{3b}$; ∴ the 4*th* or *biquadrate* root of $\frac{16a^4}{81b^4}\left(\sqrt[4]{\frac{16a^4}{81b^4}}\right)\times\frac{2a}{3b}$,

$32=2\times2\times2\times2\times2$; ∴ the *fifth* root of 32 ($\sqrt[5]{32}$) $=2$.

&c. &c.

57. If the quantity under the radical sign does not admit of resolution into the number of factors indicated by that sign, or, in other words, if it be not a *complete power*, then its exact root cannot be extracted, and the quantity itself, with the radical Sign annexed, is called a *Surd*. Thus $\sqrt{37}$, $\sqrt[3]{a^2}$, $\sqrt[4]{b^3}$, $\sqrt[5]{47}$, &c. &c. are Surd quantities. The application of the fundamental rules of arithmetic to quantities of this kind will form the subject of Chap. VIII.

58. In the involution of *negative* quantities, it was observed that the *even* powers were all +, and the *odd* powers −; there is consequently no quantity which, multiplied into itself in such manner that the number of factors shall be *even*, can generate a negative quantity. Hence quantities of the form $\sqrt{-a^2}$, $\sqrt[4]{-10}$, $\sqrt[6]{-a^3}$, $\sqrt{-5}$, $\sqrt[4]{-a^4}$, &c. &c. have no real root, and are therefore called *impossible*.

59. In extracting the roots of *compound* quantities, we must observe in what manner the terms of the *root* may be derived from those of the power. For instance, (by Art. 52,) the square of $a+b$ is $a^2+2ab+b^2$, where the terms are arranged according to the powers of a. On comparing $a+b$ with $a^2+2ab+b^2$, we observe that the first term of the power (a^2) is the square of the first term of the root (a). Put a therefore for the first term of the root, square it, and subtract that square from the first term of the power. Bring down the other two terms $2ab+b^2$, and *double* the first term of the root; set down $2a$, and having divided the first term of the remainder ($2ab$) by it, it gives b, the other term of the root; and since $2ab+b^2=(2a+b)b$, if to $2a$ the term b is added, and this sum multiplied by b, the result is $2ab+b^2$; which being subtracted from the two terms brought down, nothing remains.

$$\begin{array}{r|l}
 & a^2+2ab+b^2\,(a+b \\
 & a^2 \\
\hline
2a+b & 2ab+b^2 \\
 & 2ab+b^2 \\
\hline
 & \quad *\quad\ * \\
\end{array}$$

60. Again, the square of $a+b+c$ (Art. 54.) is $a^2+2ab+b^2+2ac+2bc+c^2$; in this case the root may be derived from the power, by continuing the process in the last Article. Thus, having found the two first terms $(a+b)$ of the root as before, we bring down the remaining three terms $2ac+2bc+c^2$ of the power, and dividing $2ac$ by $2a$, it gives c, the third term of the root. Next, let the last term (b) of the preceding divisor be doubled, and add c to the divisor thus increased, and it becomes $2a+2b+c$; multiply this new divisor by c, and it gives $2ac+2bc+c^2$, which being subtracted from the three terms last brought down, leaves no remainder. In this manner the following Examples are solved.

$$\begin{array}{r|l} & a^2+2ab+b^2+2ac+2bc+c^2\,(a+b+c \\ & a^2 \\ \hline 2a+b & 2ab+b^2 \\ & 2ab+b^2 \\ \hline 2a+2b+c & 2ac+2bc+c^2 \\ & 2ac+2bc+c^2 \\ \hline & *\quad\;\; *\quad\;\; * \end{array}$$

Ex. 1. $4x^4+6x^3+\frac{89}{4}x^2+15x+25\left(2x^2+\frac{3}{2}x+5.\right.$

$$\begin{array}{r|l} & 4x^4 \\ \hline 4x^2+\frac{3}{2}x & 6x^3+\frac{89}{4}x^2 \\ & 6x^3+\frac{9}{4}x^2 \\ \hline 4x^2+3x+5 & 20x^2+15x+25 \\ & 20x^2+15x+25 \\ \hline & *\quad\;\; *\quad\;\; * \end{array}$$

Ex. 2. $x^6+4x^5+2x^4+9x^2-4x+4\left(x^3+2x^2-x+2\right.$

$$\begin{array}{r|l} & x^6 \\ \hline 2x^3+2x^2 & 4x^5+2x^4 \\ & 4x^5+4x^4 \\ \hline 2x^3+4x^2-x & -2x^4+9x^2-4x \\ & -2x^4-4x^3+\;x^2 \\ \hline 2x^3+4x^2-2x+2 & +4x^3+8x^2-4x+4 \\ & +4x^3+8x^2-4x+4 \\ \hline & *\quad\;\; *\quad\;\; *\quad\;\; * \end{array}$$

61. The process for extracting the *Cube* Root of a compound quantity may be explained in the following manner. By Art. 52. the cube of $a+b$ is $a^3+3a^2b+3ab^2+b^3$, the terms being arranged according to the powers of a. The first

$$\begin{array}{r|l}
 & a^3+3a^2b+3ab^2+b^3(a+b \\
 & a^3 \\
\hline
3a^2+\overline{3ab+b^2} & 3a^2b+3ab^2+b^3 \\
 & 3a^2b+3ab^2+b^3 \\
\hline
 & *\quad *\quad * \\
\hline
\end{array}$$

term of the root is a, which being cubed, and this cube subtracted from the first term in the power (a^3), bring down the remaining three terms $3a^2b+3ab^2+b^3$. Next square the first term (a) of the root, and having multiplied it by 3, place $3a^2$ in the divisor, divide $3a^2b$ by $3a^2$, and it gives b the second term of the root; to $3a^2$ add $3ab+b^2$, and it forms the divisor $3a^2+3ab+b^2$, which being multiplied by b gives $3a^2b+3ab+b^3$; subtract, and nothing remains.

62. The cube root of a compound quantity, if that root consists of three terms, is found by continuing the process in a similar manner.

$$\begin{array}{r|l}
 & (a+b)^3+3(a+b)^2c+3(a+b)c^2+c^3(a+b+c \\
 & (a+b)^3 \\
\hline
3(a+b)^2+\overline{3(a+b)c+c^2} & 3(a+b)^2c+3(a+b)c^2+c^3 \\
 & 3(a+b)^2c+3(a+b)c^2+c^3 \\
\hline
 & *\quad *\quad * \\
\hline
\end{array}$$

Thus (by Art. 54) the cube of $a+b+c$ is $(a+b)^3+3(a+b)^2c+3(a+b)c^2+c^3$; supposing the first two terms of the root to have been found as in the preceding article, cube $a+b$ and subtract $(a+b)^3$ from the first term of the power; and then bring down the next three terms $3(a+b)^2c+3(a+b)c^2+c^2$. Square the two terms already found; which square being multiplied by 3, gives $3(a+b)^2$; divide $3(a+b)^2c$ by $3(a+b)^2$, and we have c, the third term of the root. To $3(a+b)^2$ add $3(a+b)c+c^2$, and it forms the divisor

$3(a+b)^2+3(a+b)c+c^2$, which being multiplied by c, gives $3(a+b)^2c+3(a+b)c^2+c^3$; subtract, and nothing remains.

XV.

On the investigation of the Rules for the Extraction of the Square and Cube Roots of Numbers.

Before we proceed to the investigation of these Rules, it will be necessary to explain the nature of the common arithmetical notation.

63. It is very well known that the value of the figures in the common arithmetical scale increases in a tenfold proportion from the right to the left; a number, therefore, may be expressed by the addition of the *units*, *tens*, *hundreds*, &c. of which it consists. Thus the number 4371 may be expressed in the following manner, viz. $4000+300+70+1$, or by $4\times1000+3\times100+7\times10+1$; hence, if the digits(a) of a number be represented by a, b, c, d, e, &c. beginning from the left hand, then,

A number of 2 figures may be expressed by $10a+b$.
. . . . . . 3 figures . . . . by $100a+10b+c$.
. . . . . . 4 figures . . . . by $1000a+100b+10c+d$.
&c. &c. &c.

54. Let a number of three figures (viz. $100a+10b+c$) be squared, and its root extracted according to the Rule in Art. 60. and the operation will stand thus;

1. $10000a^2+2000ab+100b^2+200ac+20bc+c^2(100a+10b+c$

$$\begin{array}{rll}
 & 10000a^2 & \\ \hline
200a+10b) & 2000ab+100b^2 & \\
 & 2000ab+100b^2 & \\ \cline{2-3}
 & 200a+20b+c) & 200ac+20bc+c^2 \\
 & & 200ac+20bc+c^2 \\ \cline{3-3}
 & & * \quad\quad * \quad * \\ \hline
\end{array}$$

(a) By the *digits* of a number are meant the figures which compose it, considered independently of the value which they possess in the arithmetical scale. Thus the digits of the number 537 are simply the numbers 5, 3, and 7; whereas the 5, considered with respect to its place in the numeration scale, means 500, and the 3 means 30.

II. Let $\left.\begin{array}{l} a=2 \\ b=3 \\ c=1 \end{array}\right\}$ and the operation is transformed into the following one;

$$\begin{array}{lll}
 & 40000+12000+900+400+60+1 & (200+30+1 \\
 & 40000 & \\
\hline
400+30) & 12000+900+400 & \\
 & 12000+900 & \\
\hline
 & 400+60+1 & (400+60+1 \\
 & & 400+60+1 \\
\hline
 & & *\quad *\quad * \\
\end{array}$$

III. But it is evident that this operation would not be affected by collecting the several numbers which stand in the same line into one sum, and leaving out the ciphers which are to be *subtracted* in the several parts of the operation. Let this be done; and let two figures be brought down at a time, after the square of the first figure in the root has been subtracted; then the operation may be exhibited in the manner annexed; from which it appears that the square root of 53361 is 231.

$$\begin{array}{r|l}
 & \dot{5}3\dot{3}6\dot{1}\;(231 \\
 & 4 \\
\hline
43 & 133 \\
 & 129 \\
\hline
461 & 461 \\
 & 461 \\
\hline
 & * \\
\end{array}$$

65. To explain the division of the given number into *periods* consisting of two figures each, by placing a dot over every second figure beginning with the units (as exhibited in the foregoing operation), it must be observed, that, since the square root of 100 is 10; of 10000 is 100; of 1000000 is 1000; &c. &c. it follows, that the square root of a number *less than* 100 must consist of *one* figure; of a number *between* 100 *and* 10000, of *two* figures; of a number between 10000 and 1000000, of *three* figures; &c. &c. and consequently the number of these dots will shew the number of figures contained in the square root of

the given number. From hence it also follows, that the *first* figure of the root will be the square root of the greatest square number contained in the first of those periods, reckoning from the *left*. Thus, in the case of $\dot{5}3\dot{3}6\dot{1}$ (whose square root is a number consisting of *three* figures); since the square of the figure standing in the *hundred's* place cannot be found either in the *last* period (61), or in the *last but one* (33), it must be found in the *first* period (5); consequently the first figure of the root will be the square root of the *greatest* square number contained in 5; and as this number is 4, the first figure of the root will be 2. The remainder of the operation will be readily understood by comparing the steps of it with the several steps of the process for finding the square root of $(a+b+c)^2$ in Art. 60; for having subtracted 4 for the first period (5), there remains 1; bring down the next two figures (33), and the dividend is 133; double the first figure of the root (2), and place the result 4 in the divisor; 4 is contained in 13 three times, 3 is therefore the second figure of the root; place this both in the divisor and quotient, and the former is 43; multiply by 3, and subtract 129, the remainder is 4; to which bring down the next two figures (61), which gives 461 for the next dividend. Lastly, double the last figure of the former divisor, and it becomes 46; place this in the next divisor, and since 4 is contained in 4 *once*, 1 is the third figure of the root; place 1 therefore both in the divisor and quotient; multiply and subtract as before, and nothing remains.

66. The rule for extracting the *cube* root of numbers may be understood by comparing the process for extracting the cube root of $(a+b+c)^3$ in Art^s. 61 and 62, with the following operations, in which is deduced the cube root of the number 13997521.

$$
\begin{array}{rr}
 & 1\dot{3}99\dot{7}52\dot{1} \;\left(\; 200+40+1 \right. \\
a^3=(200)^3= & \underline{8000000} \\
\text{1st Remainder} & \underline{5997521} \\
 & \\
3a^2b=3\times(200)^2\times 40= & 4800000 \\
3ab^2=3\times 200\times(40)^2= & 960000 \\
b^3=40\times 40\times 40\quad = & \underline{64000} \\
 & \underline{5824000} \\
\text{2d Remainder} & \underline{173521} \\
 & \\
3(a+b)^2c=3(200+40)^2\times 1= & 172800 \\
3(a+b)c^2=3(200+40)\;\;\times 1= & 720 \\
c^3=\qquad\quad 1\times 1\times 1= & \underline{1} \\
 & \underline{173521} \\
\text{3d Remainder} & \underline{000000}
\end{array}
$$

Omitting the superfluous ciphers, and bringing down three figures at a time, the operation would stand thus;

$$
\begin{array}{rr}
 & 1\dot{3}99\dot{7}5\dot{2}1 \;\left(\; 241 \right. \\
2^3= & \underline{8} \\
 & \underline{5997} \\
300\times 2^2\times 4= & 4800 \\
30\times 2\times 4^2= & 960 \\
4^3= & \underline{64} \\
 & \underline{5824} \\
 & \underline{173521} \\
300\times(24)^2\times 1= & 172800 \\
30\times\;24\;\;\times 1^2= & 720 \\
1^3= & \underline{1} \\
 & \underline{173521} \\
 & \underline{000000}
\end{array}
$$

These operations may be explained in the following manner.

I. Since the cube root of 1000 is 10, of 1000000 is 100, &c. it follows, that the cube root of a number less than 1000 will consist of *one* figure; of a number between 1000 and 1000000 of *two* figures, &c. &c.; if therefore the given number be divided into *periods*, each consisting of *three figures*, by placing a dot over every third figure beginning with the units, the number of those dots will shew the number of figures of which the cube root consists; and for the reason assigned in the preceding article (respecting the first figure of the square root), the *first figure* of the root will be the cube root of the greatest cube number contained in the first period.

II. Having *pointed* the number, we find that its cube root consists of *three* figures. The *first* figure is the cube root of the greatest cube number contained in 13; this being 2, the value of this figure is 200, or $a=200$; consequently $a^3=8000000$; subtract this number from 13997521, and the remainder is 5997521. Find the value of $3a^2$, and divide this latter number by it, and it gives 40 for the value of b the second member of the root; put this in the quotient, and then calculate the value of $3a^2b+3ab^2+b^3$ and subtract it, and there remains 173521. Find now the value of $3(a+b)^2$ and divide 173521 by it, and it gives 1 for the value of c the *third* member of the root; put this in the quotient, and then calculate the amount of $3(a+b)^2c+3(a+b)c^2+c^3$, which subtract, and nothing remains.

III. In reviewing the *first* of these two operations, it is evident that *six* ciphers might have been rejected in the value of a^3, and *three* in the value of $3a^2b+3ab^2+b^3$, without affecting the substance of the operation; having therefore simplified the process as in the *second* operation, we are furnished with the following Rule for extracting the cube root of numbers.

IV. "Point off every *third* figure beginning with the units; "find the greatest cube number contained in the *first* period,

“ and place the cube root of it in the quotient; *cube* it and *subtract* it from the first period, and then bring down the next three figures; divide the number thus brought down by 300 times the square of the first figure of the root, and it will give the *second* figure; then calculate the value of 300 × square of first figure × second figure + 30 × *first* figure × square of *second* + cube of second; subtract it, and then bring down the next period, and so proceed till all the periods are brought down.” The Rules for extracting the higher powers of numbers and of compound algebraic quantities are very tedious, and of no great practical utility.

XVI.

On the general mode of expressing the Powers and Roots of Quantities by means of their Indices.

67. The management of *Surd* quantities, and the method of extracting the roots of compound algebraic quantities by means of the Binomial Theorem, will be treated of in Chapters VIII, and IX; but before we conclude this Chapter, it may be proper to make a few observations on the method of expressing the powers and roots of quantities by means of their *indices*.

I. Since $a \times a^2 = a^3 = a^{1+2}$; $a^2 \times a^3 = a^5 = a^{2+3}$; or, in general, $a^m \times a^n = a^{m+n}$, it follows, that the different powers of any quantity are *multiplied* together *by adding the indices*.

II. Again, $\frac{a^2}{a} = a = a^{2-1}$; $\frac{a^5}{a^3} = a^2 = a^{5-3}$; or, in general, $\frac{a^m}{a^n} = a^{m-n}$; from which it appears, that the powers of a are *divided* by *subtracting the indices*.

III. The square of $a = a \times a = a^{1\times 2} = a^2$,
Cube of $a^2 = a^2 \times a^2 \times a^2 = a^{2\times 3} = a^6$,

or, in general, m^{th} power of $a^n = a^n \times a^n \times a^n$ to m factors $= a^{mn}$; from this it follows, that the powers of a are *raised* by multiplying the *index of a* by the *index of the power*.

IV. Square root of $a^2=a^1=a^{\frac{2}{2}}$;

Square root of $a^4=a^2=a^{\frac{4}{2}}$;

Cube root of $a^6=a^2=a^{\frac{6}{3}}$, &c. &c., i.e. the *roots* of quantities are found by *dividing* the *index of the quantity* by the *index of the root.*

68. From this method of considering the formation of the powers and roots of quantities, a new species of algebraic notation arises, of which the following are Examples.

I. The *roots* of quantities may be expressed by *fractional indices.* Thus,

The square root of $a=a^{1\div 2}=a^{\frac{1}{2}}$;

Cube root of $a=a^{1\div 3}=a^{\frac{1}{3}}$;

or, in general, m^{th} root of $a=a^{1\div m}=a^{\frac{1}{m}}$.

Again, cube root of $a^2=a^{2\div 3}=a^{\frac{2}{3}}$;

Square root of $a^3=a^{3\div 2}=a^{\frac{3}{2}}$;

5th root of $a^2=a^{2\div 5}=a^{\frac{2}{5}}$;

or, in general, m^{th} root of $a_n=a^{n\div m}=a^{\frac{n}{m}}$.

II. Any quantity raised to the power of 0 is equal to unity.

Thus, $\frac{a^m}{a^n}=a^{m-n}=$ (if $n=m$) $a^{m-m}=a^0$;

But $\frac{a^m}{a^m}=1$;

Hence $a^0=1$.

III. The powers and roots of quantities may be expressed by *negative* indices.

For $\frac{a^m}{a^n}=a^{m-n}=$(if $m=0$) a^{-n};

but $\frac{a^0}{a^n}=\frac{1}{a^n}$;

Hence $\frac{1}{a^n}=a^{-n}$, and, *vice versa*, $a^n=\frac{1}{a^{-n}}$.

IV. **From this it follows, that any quantity may be removed from the *numerator* of a fraction into the *denominator*, or from the *denominator* into the *numerator*, by *changing the sign* of its index.**

Ex. 1. Thus (since $\frac{1}{b^3}=b^{-3}$) $\frac{a^2}{b^3}$ may be expressed by a^2b^{-3}; and $\left(\text{since } a^2=\frac{1}{a^{-2}}\right)$, we have $\frac{a^2}{b^3}=\frac{1}{a^{-2}}\times\frac{1}{b^3}=\frac{1}{a^{-2}b^3}$.

Ex. 2. The quantity $\frac{a^2b^3}{c^2d^4e^5}$ may be expressed by $a^2b^3c^{-2}d^{-4}e^{-5}$, or by $\frac{1}{a^{-2}b^{-3}c^2d^4e^5}$.

CHAP. IV.

ON SIMPLE EQUATIONS.

When two algebraic quantities are connected together by the sign of equality, the whole expression thus formed is called (Def. 11.) an *Equation.* Equations, as applied to the solution of questions or problems, consist of quantities, some of which are *known*, and others *unknown;* and by the *solution* of an equation is meant, the operation by which the value of the unknown quantities are found in terms of the known ones. If an equation contains no *power* of the unknown quantities, but those quantities merely in their simplest form, it is called a *Simple* Equation; if it contains the *square* of the unknown quantity, it is called a *Quadratic* Equation; if the *cube* of the unknown quantity, a *Cubic* Equation; &c. &c. The present Chapter will be occupied entirely with the solution of *Simple* Equations, and questions depending upon them.

XVII.

On the Solution of Simple Equations, containing only one unknown quantity.

69. The rules absolutely necessary for the solution of simple equations containing only one unknown quantity may be reduced to four, and may be arranged in the following order.

Rule I.

The first rule is, that " any quantity may be transferred from " one side of the equation to the other, by changing its sign;

and it is founded upon the axiom, that "if equals be *added* to "or *subtracted* from equals, *the sums* or *remainders* will be "equal."

Ex. 1. Let $x+8=15$; *subtract* 8 from each side of the equation, and it becomes $x+8-8=15-8$; but $8-8=0$, $\therefore x=15-8=7$.

Ex. 2. Let $x-7=20$; *add* 7 to each side of the equation, then $x-7+7=20+7$; but $-7+7=0$, $\therefore x=20+7=27$.

Ex. 3. Let $3x-5=2x+9$; *add* 5 to each side of the equation, and it becomes $3x-5+5=2x+9+5$, or $3x=2x+9+5$. *Subtract* $2x$ from each side of this latter equation, then $3x-2x=2x-2x+9+5$; but $2x-2x=0$, $\therefore 3x-2x=9+5$. Now $3x-2x=x$, and $9+5=14$; hence $x=14$.

On reviewing the steps of these examples, it appears

I. That $x+8=15$ is identical with . . . . $x=15-8$.

II. . . . $x-7=20$ . . . . . . with . . . . $x=20+7$.

III. . . $3x-5=2x+9$ . . . . with $3x-2x=9+5$.

Or, that "the equality of the quantities on each side of the "equation, is not affected by removing a quantity from one "side of the equation to the other and *changing its sign*."

From this Rule also it appears, if the same quantity with the same sign be found on *both* sides of an equation, it may be left out of the equation; thus, if $x+a=c+a$, then $x=c+a-a$; but $a-a=0$, $\therefore x=c$.

It further appears, that the signs of all the terms of an equation may be changed from + to −, or from − to +, without altering the value of the unknown quantity. For let $x-b=c-a$; then, by the Rule, $x=c-a+b$; change the signs of *all* the terms, then $b-x=a-c$, in which case $b-a+c=x$, or $x=c-a+b$, as before.

RULE II.

"If the unknown quantity has a coefficient, then its value "may be found by dividing each side of the equation by that "coefficient;" and the foundation of the Rule is, that "if equals "be divided by the same, the quotients arising will be equal."

Ex. 1. Let $2x=14$; then *dividing* both sides of the equation by 2, we have $\frac{2x}{2}=\frac{14}{2}$; but $\frac{2x}{2}=x$, and $\frac{14}{2}=7$, $\therefore x=7$.

Ex. 2. Let $6x+10=3x+22$; then, by RULE I, $6x-3x=22-10$, or $3x=12$; *divide* each side by 3, then $\frac{3x}{3}=\frac{12}{3}$, or $x=4$.

Ex. 3. Let $ax=b+c$; then $\frac{ax}{a}=\frac{b+c}{a}$; but $\frac{ax}{a}=x$; $\therefore x=\frac{b+c}{a}$.

RULE III.

"An equation may be cleared of fractions, by multiplying each "side of the equation by the denominators of the fractions in "succession, or by their least common multiple." This Rule goes upon the principle, that "if equals be multiplied by the "same, the products arising will be equal."

Ex. 1. Let $\frac{x}{3}=6$; *multiply* each side of the equation by 3, then (since, from what has been already shewn, the multiplication of the fraction $\frac{x}{3}$ by 3, just takes away its denominator, and gives x) we have $x=6\times3=18$.

Ex. 2. Let $\frac{x}{2}+\frac{x}{5}=7$; *multiply* each side of the equation by 2, and we have $x+\frac{2x}{5}=14$; now multiply each side by 5, and it becomes $5x+2x=70$, or $7x=70$; hence, by RULE II, $x=\frac{70}{7}=10$.

Ex. 3. Let $\frac{x}{2}+\frac{x}{3}=13-\frac{x}{4}$.

Multiply each side of the equation by 2, then $x+\frac{2x}{3}=26-\frac{2x}{4}$

. . . . by 3, and $3x+2x=78-\frac{6x}{4}$

. . . . by 4 . . . $12x+8x=312-6x$

by RULE I, $12x+8x+6x=312$

or $26x=312$

$\therefore$ by RULE II, $x=\frac{312}{26}=12$

This Example might have been solved more simply, by multiplying each side of the equation by the *least common multiple* of the numbers 2, 3, 4, which is 12.

Thus, $\frac{x}{2}+\frac{x}{3}=13-\frac{x}{4}$.

Multiply each side by 12, $\frac{12x}{2}+\frac{12x}{3}=156-\frac{12x}{4}$,

or $6x+4x=156-3x$;

$\therefore$ by RULE I, $6x+4x+3x=156$

or $13x=156$, and $x=\frac{156}{13}=12$.

RULE IV.

"If the equation contains the square root of the unknown "quantity, or the square root of the unknown quantity combined "with some known quantity; then, let this surd quantity be "brought by itself to one side of the equation, and let both "sides of the equation be *squared*; the value of the unknown "quantity may then be found by the preceding Rules." This Rule goes upon the supposition, that "if the *square root* of "a quantity be equal to any given quantity, then the *quantity* "*itself* will be equal to the *square* of that given quantity."

Ex. 1. Let $\sqrt{x}-5=3$; then by RULE I, $\sqrt{x}=5+3=8$; *square* both sides of the equation, then $x=8\times 8=64$.

Ex. 2. Let $\sqrt{2x+1}+2=5$; then, by RULE I, $\sqrt{2x+1}=5-2=3$; *square* both sides of the equation, and we have $2x+1=9$, $\therefore 2x=9-1=8$, and $x=\frac{8}{2}=4$.

70. The following Examples will serve to exercise the learner in these several Rules.

In RULE I.

✓Ex. 1. $2x+3=x+17$ . . . ANSWER, $x=14$.

Ex. 2. $5x-4=4x+25$ . . . . $x=29$.

✓Ex. 3. $7x-9=6x-3$ . . . . $x=6$.

✓Ex. 4. $4x+2a=3x+9b$ . . . . $x=9b-2a$.

In RULES I, and II.

Ex. 1. $\frac{2x}{3}+\frac{x}{4}=22$ . . . . ANSWER, $x=24$.

Ex. 2. $\frac{7x}{4}-\frac{5x}{6}=\frac{55}{6}$ . . . . $x=10$.

Ex. 3. $\frac{x}{2}+\frac{x}{3}=31-\frac{x}{5}$ . . . . $x=30$.

Ex. 4. $\frac{2x}{5}-\frac{x}{6}+\frac{x}{2}=44$ . . . . $x=60$.

In RULES I, II, III.

Ex. 1. $10x=150$ . . . . ANSWER, $x=15$.

Ex. 2. $15x+4=34$ . . . . $x=2$.

Ex. 3. $8x+7=6x+27$ . . . . $x=10$.

Ex. 4. $9x-3=4x+22$ . . . . $x=5$.

Ex. 5. $17x-4x+9=3x+39$ . . . . $x=3$.

Ex. 6. $ax-c=b+2c$ . . . . $x=\frac{b+3c}{a}$.

In RULE IV.

Ex. 1. $\sqrt{x}-1=4$ . . . . ANSWER, $x=25$

Ex. 2. $\sqrt{3x+1}+5=10$ . . . . $x=8$.

Ex. 3. $15+\sqrt{x+7}=19$ . . . . $x=9$.

71. In the application of these Rules to the solution of simple equations in general containing only one unknown quantity, it will be proper to observe the following method.

I. To clear the equation of fractions by RULE III.

II. To collect the *unknown* quantities on one side of the equation, and the *known* on the other, by RULE I.

III. To find the value of the unknown quantity by dividing each side of the equation by its coefficient, as in RULE II.

IV. If the equation contains a *surd* quantity, then RULE IV. must be immediately applied.

EXAMPLE 1.

Find the value of x in the equation $\frac{3x}{7}+1=\frac{x}{5}+\frac{13}{5}$.

Multiply by 7, then $3x+7=\frac{7x}{5}+\frac{91}{5}$;

. . . . . by 5, . . $15x+35=7x+91$.

Collect the unknown quantities on *one* side, and the known on the other; $15x-7x=91-35$,

or $8x=56$.

Divide by the coefficient of x, $x=\frac{56}{8}=7$.

Ex. 2.

Find the value of x in the equation $\frac{x+3}{5}-1=2-\frac{x}{7}$.

Multiply by 5, then $x+3-5=10-\frac{5x}{7}$;

. . . . . by 7 . . . $7x+21-35=70-5x$.

Collect the *unknown* quantities on *one* side, and the known on the *other;* $7x+5x=70-21+35$,

or $12x=84$;

$\therefore x=\frac{84}{12}=7$.

Ex. 3.

Ex. 3.

Find the value of x in the equation

$$4x-\frac{x-1}{2}=x+\frac{2x-2}{5}+24.$$

Multiply by the *least* common multiple (10), $\Big\}\ 40x-5x+5=10x+4x-4+240$(a).

By transposition, $40x-5x-10x-\ 4x=240-4-5.$

$$\text{or } 40x-19x=231,$$
$$\text{i.e. } 21x=231;$$
$$\therefore x=\frac{231}{21}=11.$$

Ex. 4.

Find the value of x in the equation $2x-\frac{x}{2}+1=5x-2.$

Multiply by 2, then $4x-x+2=10x-4.$

By transposition, $4+2=10x-4x+x,$

$$\text{or } 6=\ 7x;$$
$$\therefore \frac{6}{7}=x.$$
$$\text{or } x=\frac{6}{7}.$$

Ex. 5.

What is the value of x in the equation $3ax+2bx=3c+a$?

Here $3ax+2bx=(3a+2b)\times x;$

$\therefore (3a+2b)\times x=3c+a.$

Divide each side of the equation by $3a+2b$, which is the coefficient of x; then $x=\dfrac{3c+a}{3a+2b}.$

Ex. 6.

(a) As this step involves the case "where the sign − stands before a Fraction," when the numerator of that fraction is brought down into the same line with $40x$, the signs of both its terms must be *changed*, for the reasons assigned in Ex. 3, page 28; and we therefore make it $-5x+5$, and not $5x-5$.

Ex. 6.

Find the value of x in the equation $3bx+a=2ax+4c$. Bring the *unknown* quantities to *one* side of the equation, and the *known* to the *other;* then,

$$3bx-2ax=4c-a,$$

$$\text{but } 3bx-2ax=(3b-2a)\times x;$$

$$\therefore (3b-2a)x=4c-a.$$

Divide by $3b-2a$, and $x=\frac{4c-a}{3b-2a}$.

Ex. 7.

Find the value of x in the equation $bx+x=2x+3a$. Transpose $2x$, then $bx+x-2x=3a$,

$$\text{or } bx-\ x=3a,$$

$$\text{but } bx-\ x=(b-1)x;$$

$$\therefore (b-1)x=3a,$$

$$\text{and } x=\frac{3a}{b-1}.$$

Ex. 8.

Find the value of x in the equation $\frac{3x}{a}-c+\frac{x}{b}=4x+\frac{2x}{d}$.

Multiply by abd, then $3bdx-abcd+adx=4abdx+2abx$. By transposition, $3bdx+adx-4abdx-2abx=abcd$,

$$\text{or } (3bd+ad-4abd-2ab)x=abcd$$

$$\therefore x=\frac{abcd}{3bd+ad-4abd-2ab}.$$

Ex. 9.

Let $\sqrt{x}+\sqrt{a+x}=\frac{2a}{\sqrt{a+x}}$ to find the value of x.

Multiply by $\sqrt{a+x}$, then $\sqrt{x}\times\sqrt{a+x}+a+x=2a$.

By *transposition,* $\sqrt{x}\times\sqrt{a+x}=2a-a-x=a-x$.

Square both sides, $x\times(a+x)=a^2-2ax+x^2$,

$$\text{or } ax+x^2=a^2-2ax+x^2;$$

$$\therefore 3ax=a^2$$

$$\text{and } x=\frac{a^2}{3a}=\frac{a}{3}.$$

Ex. 10.

Let $a+x=\sqrt{a^2+x\sqrt{b^2+x^2}}$ to find the value of x.

Square both sides, and we have $a^2+2ax+x^2=a^2+x\sqrt{b^2+x^2}$,

$$\text{or } 2ax+x^2=x\sqrt{b^2+x^2}.$$

Divide by x, $2a+x=\sqrt{b^2+x^2}$.

Square *again*, $4a^2+4ax+x^2=b^2+x^2$;

$$\therefore 4a^2+4ax=b^2,$$

$$\text{or } 4ax=b^2-4a^2.$$

Hence, $x=\dfrac{b^2-4a^2}{4a}=\dfrac{b^2}{4a}-a$.

Ex. 11. $x+\dfrac{x}{2}+\dfrac{x}{3}=11$ . . . ANSWER, $x=6$.

Ex. 12. $\dfrac{x}{5}+\dfrac{x}{4}+\dfrac{x}{3}=\dfrac{x}{2}+17$ . . . . . . $x=60$.

Ex. 13. $4x-20=\dfrac{3x}{7}+\dfrac{110}{7}$ . . . . . . $x=10$.

Ex. 14. $\dfrac{x}{2}+\dfrac{x}{3}-\dfrac{x}{4}=\dfrac{1}{2}$ . . . . $x=\dfrac{6}{7}$.

Ex. 15. $3x+\dfrac{1}{9}=\dfrac{x+3}{3}$ . . . . $x=\dfrac{1}{3}$.

Ex. 16. $\dfrac{3x}{7}-5=29-2x$ . . . . $x=14$.

Ex. 17. $6x-\dfrac{3x}{4}-9=5x$ . . . . $x=36$.

Ex. 18. $2x-\dfrac{x+3}{3}+15=\dfrac{12x+26}{5}$ . . $x=12$.

Ex. 19. $\dfrac{x-2}{2}+\dfrac{x}{3}=20-\dfrac{x-6}{2}$ . . . . . . $x=18$.

Ex. 20. $5x-\dfrac{2x-1}{3}+1=3x+\dfrac{x+2}{2}+7$ . . $x=8$.

Ex. 21. $2ax+b=3cx+4a$ . . . . . . $x=\dfrac{4a-b}{2a-3c}$.

Ex. 22. $5ax-2b+4bx=2x+5c$ .... $x=\frac{5c+2b}{5a+4b-2}$.

Ex. 23. $bx+2x-a=3x+2c$ ... $x=\frac{2c+a}{b-1}$.

Ex. 24. $3x-a+cx=\frac{a+x}{3}-\frac{b-x}{a}$ .. $x=\frac{4a^2-3b}{8a+3ac-3}$.

XVIII.

On the Solution of Simple Equations containing Two or more unknown Quantities.

For the solution of equations containing two or more unknown quantities, as many independent equations are required as there are unknown quantities. The two equations necessary for the solution of the case when *two* unknown quantities are concerned may be expressed in the following manner,

$$ax+by=c$$
$$a'x+b'y=c'$$

where a, b, c, a', b', c' represent *known* quantities, and x, y the *unknown* quantities whose values are to be found in terms of these known quantities.

72. There are three different Rules by which the value of one of these unknown quantities may be determined;

Rule 1.

Let $ax+by=c$ (A) and $a'x+b'y=c'$ (B) } be the two equations to be solved.

Multiply equation (A) by a', then $aa'x+a'by=a'c$ (C)

. . . . (B) by a, ... $aa'x+ab'y=ac'$ (D)

Subtract equation (D) from (C), then $(a'b-ab')y=a'c-ac'$

$$\therefore y=\frac{a'c-ac'}{a'b-ab'}.$$

From which we deduce the following Rule. " Multiply the " first equation by the coefficient of x in the second equation,

" and the second equation by the coefficient of x in the first " equation; subtract the *last* of these resulting equations from " the *first*, and there will arise an equation which contains only " y and known quantities, from which the value of y is determined."

Rule II.

From equation (A), $ax=c-by$, $\therefore x=\frac{c-by}{a}$

. . . . (B), $a'x=c'-b'y$, $\therefore x=\frac{c'-b'y}{a'}$

Putting these two values of x equal to each other, we have

$$\frac{c'-b'y}{a'}=\frac{c-by}{a}; \text{ and } \therefore\ ac'-ab'y=a'c-a'by;$$

$$\text{By transposition, } (a'b-ab')y=a'c-ac'$$

$$\text{and } y=\frac{a'c-ac'}{a'b-ab'}.$$

From which it appears, that "if the value of x in the *first* " equation be put equal to its value in the *second*, there will " arise a new equation involving only y, from which the same " value of y is found as before."

Rule III.

From equation (A), $x=\frac{c-by}{a}$; substitute this value of x in equation (B), then $a'\times\frac{c-by}{a}+b'y=c'$

$$\text{or } a'c-a'by+ab'y=ac'$$

$$\therefore a'c-ac'=(a'b-ab')y$$

$$\text{and } y=\frac{a'c-ac'}{a'b-ab'}.$$

From which we infer, that " if the value of x found from the " *first* equation be substituted for it in the *second*, there will " arise an equation which gives the same value of y as in the " two former instances."

73. Having determined the value of y, the value of x may be found in each case, by substituting this value for y either in the first or second equation. The value of x in the first equation is $\frac{c-by}{a}$; but $y=\frac{a'c-ac'}{a'b-ab'}$, $\therefore\ x=\frac{c}{a}-\frac{b(a'c-ac')}{a(a'b-ab')}$ =(by reducing these fractions to a common denominator) $\frac{bc'-b'c}{a'b-ab'}$.

The value of x in the *second* equation is $\frac{c'-b'y}{a'}=\frac{c'}{a'}-\frac{b'(a'c-ac')}{a(a'b-ab')}$ =(by reducing these fractions to a common denominator) $\frac{bc'-b'c}{ab-ab'}$ as before.

74. From hence it appears, that in finding the value of y, *either* of the three Rules may be applied; and that in finding the value of x, the value of y so found may be substituted either in the *first* or *second* equation. In the choice of the Rule which may be most adapted to practical application, experience only can be our guide. It may further be observed, that there are cases in which RULE I. may be somewhat varied; for instance, if the given equations be,

$$ax+by=c\ \ (A)$$
$$a'x-b'y=c'\ \ (B)$$

Multiply equation (A) by b', then, $ab'x+bb'y=b'c$ (C)
. . . . (B) by b, ... $a'bx-bb'y=bc'$ (D)

Add equation (D) to (C), then $(ab'+a'b)x=b'c+bc'$

$$\text{and } x=\frac{b'c+bc'}{ab'+a'b}.$$

Having the value of x, the value of y may be found by some of the preceding methods.

75. The following examples are intended to illustrate each Rule separately.

Ex. 1.

EXAMPLE I.

$$\left.\begin{array}{r} \text{Let } 5x+4y=55\ (A) \\ 3x+2y=31\ (B) \end{array}\right\}\text{ to find the values of } x \text{ and } y.$$

By RULE I,

Multiply (A) by 3, then $15x+12y=165$

. . . . . (B) by 5 . . . $15x+10y=155$

$\therefore$ by *subtraction*, we have $2y=10$, or $y=\frac{10}{2}=5$.

Now from equation (A) we have $x=\frac{55-4y}{5}=$ (since $y=5$, and $\therefore 4y=20$) $\frac{55-20}{5}=\frac{35}{5}=7$.

Ex. 2.

$$\left.\begin{array}{r} \text{Let } x+4y=16\ (A) \\ 4x+\ \ y=34\ (B) \end{array}\right\}$$

From equation (A), we have $x=16-4y$.

. . . . (B) . . . . . $x=\frac{34-y}{4}$.

Hence, by RULE II, $\frac{34-y}{4}=16-4y$,

or $34-y=64-16y$;

$\therefore 15y=30$ or $y=\frac{30}{15}=2$.

It has already been shewn that $x=16-4y=$ (since $y=2$, and $\therefore 4y=8$) $16-8=8$.

Ex. 3.

$$\text{Let } \frac{x+2}{3}+8y=31\ \ (A)$$

$$\frac{y+5}{4}+10x=192\,(B).$$

Clear eq^{ns}. (A) of fract^{s}. $x+2+24y=\ 93$, or $x+24y=\ 91\ (C)$

. . . . . (B) . . . . . $y+5+40x=768$, or $y+40x=763\ (D)$.

From equation (C), $x=91-24y$; by RULE III, substitute this value of x in equation (D); then we have,

$$y+40(91-24y)=763,$$
$$\text{or } y+3640-960y=763$$
$$\therefore\ 959y=3640-763=2877$$
$$\text{and } y=\frac{2877}{959}=3.$$

By referring to equation (C), we have $x=91-24y=$ (since $y=3$, and $\therefore$ $24y=72$) $91-72=19$.

Ex. 4.

$$\text{Let } 3x+4y=29\ (A).$$
$$17x-3y=36\ (B).$$

In this example, the Rule mentioned in Art. 74. must be applied thus;

Multiply equation (A) by 3, then $9x+12y=\ 87\ (C)$

. . . . (B) by 4 . . . $\underline{68x-12y=144\ (D)}$

Add equation (D) to (C), then $77x=231$ or $x=\frac{231}{77}=3$.

From equation (A) we have $4y=29-3x=$ (since $x=3$, and $\therefore$ $3x=9$) $29-9=20$; hence $y=\frac{20}{4}=5$.

Ex. 5. $\left.\begin{array}{l}4x+3y=31\\3x+2y=22\end{array}\right\}$ . . . . . . ANSWER, $\left\{\begin{array}{l}x=4\\y=5.\end{array}\right.$

Ex. 6. $\left.\begin{array}{l}3x+2y=40\\2x+3y=35\end{array}\right\}$ . . . . $\left\{\begin{array}{l}x=10\\y=5.\end{array}\right.$

Ex. 7. $\left.\begin{array}{l}5x-4y=19\\4x+2y=36\end{array}\right\}$ . . . . $\left\{\begin{array}{l}x=7\\y=4.\end{array}\right.$

Ex. 8. $\left.\begin{array}{l}3x+7y=79\\2y-\frac{1}{2}x=\ 9\end{array}\right\}$ . . . . $\left\{\begin{array}{l}x=10\\y=\ 7.\end{array}\right.$

Ex. 9. $\left.\begin{array}{l}\frac{x+y}{3}+1=\ 6\\[1ex]\frac{x-y}{7}+3=\ 4\end{array}\right\}$ . . . . $\left\{\begin{array}{l}x=11\\y=\ 4.\end{array}\right.$

Ex.

$$\text{Ex. 10.}\quad \left.\begin{array}{l} \frac{x+y}{3}-2y=2 \\ \frac{2x-4y}{5}+y=\frac{23}{5} \end{array}\right\} \ldots\ldots \text{ANSWER,} \left\{\begin{array}{l} x=11 \\ y=\ 1. \end{array}\right.$$

$$\text{Ex. 11.}\quad \left.\begin{array}{l} \frac{2x-3}{2}+y=7 \\ 5x-13y=\frac{67}{2} \end{array}\right\} \ldots\ldots\ldots\ldots \left\{\begin{array}{l} x=8 \\ y=\frac{1}{2}. \end{array}\right.$$

$$\text{Ex. 12.}\quad \left.\begin{array}{l} \frac{3x-7y}{3}=\frac{2x+y+1}{5} \\ 8-\frac{x-y}{5}=6 \end{array}\right\} \ldots\ldots\ldots\ldots \left\{\begin{array}{l} x=13 \\ y=\ 3. \end{array}\right.$$

76. When *three* unknown quantities are concerned, the most general form under which equations of this kind can be expressed, is

$$ax+by+cz=d \ (E)$$
$$a'x+b'y+c'z=d' \ (F)$$
$$a''x+b''y+c''z=d'' \ (G),$$

and the mode of solution may be conducted in the following manner.

I. Multiply eq^n. (E) by a', then $aa'x+a'by+a'cz=a'd$ (H)

. . . . (F) by a . . . $aa'x+ab'y+ac'z=ad'$ (K)

Subtract (K) from (H), then $(a'b-ab')y+(a'c-ac')z=a'd-ad'$ (L).

By multiplying (F) by a'', and (G) by a', and subtracting the latter result from the former, we obtain in the same manner, $(a''b'-a'b'')y+(a''c'-a'c'')z=a''d'-a'd''$ (M).

II. Next, let the known quantities in equation (L) be represented by α, β, γ respectively; and those in equation (M) by α', β', γ' respectively; then those equations may be reduced to the following form, viz.

$\alpha y+$

$$\alpha y+\beta z=\gamma.$$

$$\alpha' y+\beta' z=\gamma'.$$ From which, by making the proper substitutions in RULE I, and in Art. 73, we have,

$$z=\frac{\alpha'\gamma-\alpha\gamma'}{\alpha'\beta-\alpha\beta'}.$$

$$y=\frac{\beta\gamma'-\beta'\gamma}{\alpha'\beta-\alpha\beta'}.$$

III. From equation (E), we have $x=\frac{d}{a}-\frac{by+cz}{a}$; in which, substituting the values of y and z just now found, we obtain

$$x=\frac{d}{a}-\frac{b(\beta\gamma'-\beta'\gamma)+c(\alpha'\gamma-\alpha\gamma')}{a(\alpha'\beta-\alpha\beta')}.$$

This mode of operation might be easily extended to equations containing any number of unknown quantities.

EXAMPLE I.

$$\left.\begin{array}{l}\text{Let } 2x+3y+4z=29\ (E)\\ \quad\ \ 3x+2y+5z=32\ (F)\\ \quad\ \ 4x+3y+2z=25\ (G)\end{array}\right\}\text{to find the values of } x, y, z.$$

I. Multiply (E) by 3, then $6x+9y+12z=87\ (H)$

. . . . . (F) by 2 . . . $6x+4y+10z=64\ (K)$.

Subtract (K) from (H) . . $5y+\ 2z=23\ (L)$.

Multiply (F) by 4, then $12x+8y+20z=128$

. . . . . (G) by 3, . . . $12x+9y+\ 6z=75$

Subtract . . . . . $-y+14z=53\ (M)$.

II. Hence the given equations are reduced to,

$$5y+\ 2z=23\ (L)$$

$$-y+14z=53\ (M).$$

Again . . . . $5y+\ 2z=23$

Multiply (M) by 5, then $-5y+70z=265$

By *addition* . . . . . . $72z=288$, or $z=\frac{288}{72}=4$

From equation (M) . . . . $y=14z-53=56-53=3$.

III. From equation (E) . . $x=\frac{29-3y-4z}{2}=\frac{29-25}{2}=2.$

Ex. 2. $$\left.\begin{array}{r} x+y+z=90\\ 2x+40=3y+20\\ 2x+40=4z+10 \end{array}\right\} \ldots \text{ ANSWER, } \left\{\begin{array}{l} x=35\\ y=30\\ z=25. \end{array}\right.$$

Ex. 3. $$\left.\begin{array}{r} x+y+z=53\\ x+2y+3z=105\\ x+3y+4z=134 \end{array}\right\} \ldots\ldots\ldots \left\{\begin{array}{l} x=24\\ y=6\\ z=23. \end{array}\right.$$

XIX.

The Solution of Questions producing Simple Equations.

In the reduction and management of equations, we have proceeded by fixed and stated rules; but in the solution of *questions* we have no such rules to guide us. Every particular question requires a distinct process of reasoning, to bring it into an algebraic form; and nothing but practice and experience can produce expertness and facility in effecting this process. All that can be done for the learner in this case, is, to explain the manner in which the principles of this science may be made to bear upon questions in general; for as soon as they can be brought into the shape of *equations*, we have only to apply the foregoing Rules for finding the value of the unknown quantity or quantities. Before we proceed, therefore, to any actual examples, it may be proper to shew the relation which arithmetical and algebraic operations stand in to each other.

77. Suppose the following arithmetical question was proposed for solution; viz. "To divide the number 35 into two such "parts, that one part should exceed the other part by 9." A person unacquainted with Algebra might with no great difficulty solve this question in the following manner.

I. It appears, in the first place, that there must be a *greater* and a *lesser* part.

II. The greater part must exceed the lesser by 9.

III. But it is evident that the greater and lesser parts added together must be equal to the whole number 35.

IV. If then we substitute for the greater part its *equivalent*, viz. "*the lesser part increased by* 9"," it follows, that the lesser part increased by 9, with the *addition* of the said lesser part, is equal to 35.

V. Or, in other words, that *twice* the lesser part with the addition of 9, is equal to 35.

VI. Therefore, *twice the lesser part* must be equal to 35, *with* 9 *subtracted from it.*

VII. Hence, twice the lesser part is equal to 26.

VIII. From which we conclude, that the *lesser part* is equal to 26 *divided by* 2; i.e. to 13.

IX. And consequently, as the *greater* part exceeds the *lesser* by 9, it must be equal to 22.

But by adopting the method of algebraic *notation, the different steps of this solution may be much more briefly expressed as follows.*

I. Let the *lesser* part . . . . $=x$.
II. Then the *greater* part . . . . $=x+9$.
III. But greater part + lesser part . . . . $=35$.
IV. $\therefore$ $x+9+x$ . . . . $=35$.
V. or $2x+9$ . . . . $=35$.
VI. $\therefore$ $2x$ . . . . $=35-9$.
VII. or $2x$ . . . . $=26$.
VIII. $\therefore$ x (*lesser* part) . . . . $=\frac{26}{2}=13$.
IX. and $x+9$ (*greater* part) . . . . $=13+9=22$.

78. Having thus explained the manner in which the several steps in the solution of an arithmetical question may be expressed in the language of Algebra, we now proceed to its exemplification.

QUESTION 1.

There are two numbers whose difference is 15 and their sum 59. What are the numbers?

As their *difference* is 15, it is evident that the greater number must exceed the lesser by 15.

Let, therefore, $x=$ the lesser number
then will $x+15=$ the greater
But their *sum* $=59$
$\therefore\ x+x+15=59$
or $2x+15=59$
and $2x=59-15=44$
$\therefore\ x=\frac{44}{2}=22$ the lesser number
and $x+15=22+15=37$ the greater.

QUESTION 2.

What two numbers are those whose difference is 9; and if three times the greater be added to five times the lesser, the sum shall be 35?

Let $x=$ the *lesser* number;
then $x+9=$ *greater* number.
And 3 *times* the greater $=3\times(x+9)=3x+27$,
5 *times* the lesser $=5x$.
But by the question, 3 times the greater + 5 times the lesser $=35$.
Hence, $(3x+27)+(5x)\ .\ .\ .\ .\ .\ .\ =35$,
$\therefore\ 8x+27=35$,
or $8x=35-27=8$; $\therefore\ x=1$ *lesser* number,
and $x+9=1+9=10$ the *greater* number.

QU. 3.

QUESTION 3.

What number is that to which 10 being added, $\frac{3}{5}$ths of the sum shall be 66?

Let x = the number required;

then $x+10$ = the number, with 10 added to it.

Now $\frac{3}{5}$ths of $(x+10)=\frac{3}{5}(x+10)=\frac{3(x+10)}{5}=\frac{3x+30}{5}$.

But, by the question, $\frac{3}{5}$ths of $(x+10)=66$;

$$\text{Hence, } \frac{3x+30}{5}=66.$$

Multiply by 5, then $3x+30=330$;

$$\therefore 3x=330-30=300; \text{ or } x=\frac{300}{3}=100.$$

QUESTION 4.

What number is that which being multiplied by 6, the product increased by 18, and that sum divided by 9, the quotient shall be 20?

Let x = the number required;

then $6x$ = the number multiplied by 6;

$6x+18$ = the product increased by 18,

and $\frac{6x+18}{9}$ = that sum divided by 9.

Hence, by the question, $\frac{6x+18}{9}=20$.

Multiply by 9, then $6x+18=180$,

or $6x=180-18=162$; or $x=\frac{162}{6}=27$.

QUESTION 5.

A post is $\frac{1}{5}$th in the earth, $\frac{3}{7}$ths in water, and 13 feet out of the water. What is the length of the post?

Let x = length of the post;

then $\frac{x}{5}$ = the part of it in the earth,

$\frac{3x}{7}$ = the part of it in the water,

13 = the part of it out of the water.

But part in earth + part in water + part out of water = whole post;

$$\therefore \left(\frac{x}{5}\right) + \left(\frac{3x}{7}\right) + 13 = x.$$

Multiply by 5, then $x+\frac{15x}{7}+65=5x$;

. . . . . by 7 . . $7x+15x+455=35x$,

or $455=35x-7x-15x=13x$.

Hence $x=\frac{455}{13}=35$ length of post.

Question 6.

After paying away $\frac{1}{4}$th and $\frac{1}{7}$th of my money, I had £.85 left in my purse; What money had I at first?

Let x = money in purse at first;

then $\frac{x}{4}+\frac{x}{7}$ = money paid away.

But money at first − money paid away = money remaining.

$$\text{Hence } x - \left(\frac{x}{4}+\frac{x}{7}\right) = 85,$$

$$\text{i.e. } x-\frac{x}{4}-\frac{x}{7} = 85.$$

Multiply by 4, then $4x-x-\frac{4x}{7}=340$;

. . . . . by 7 . . $28x-7x-4x=2380$,

$\therefore 17x=2380$; or $x=\frac{2380}{17}=140l.$

Question 7.

Of a battalion of soldiers, $\frac{3}{4}$th are on duty, $\frac{1}{10}$th are sick, $\frac{2}{5}$ths of the remainder are absent, and there are 48 officers. What is the number of persons in the battalion?

Let x = the number of persons in the battalion.

Then $\frac{3}{4}$ths of x, or $\frac{3x}{4}$, = men on duty,

$\frac{1}{10}$th of x, or $\frac{x}{10}$, = the sick;

And $\frac{3x}{4}+\frac{x}{10}$, or $\frac{34x}{40}$, $=\frac{17x}{20}=$ men on duty and sick.

$$\text{Hence } x-\frac{17x}{20}=\frac{3x}{20}=\text{remainder,}$$

And $\frac{3}{5}$ths of $\frac{3x}{20}$, or $\frac{9x}{100}$, $=\frac{3}{5}$ths of remainder $=$ the absent.

But the men on *duty*, the *sick*, the *absent*, and the officers together, make up the *whole battalion;*

$$\text{i. e. } \frac{17x}{20}+\frac{9x}{100}+48=x;$$

$$\text{or } 17x+\frac{9x}{5}+960=20x;$$

$$\therefore 85x+9x+4800=100x.$$

$$\text{Hence } 100x-85x-9x=4800,$$

$$\text{or } 6x=4800; \text{ or } x=\frac{4800}{6}=800.$$

QUESTION 8.

There are two numbers, such, that 3 times the greater added to $\frac{1}{3}$d the lesser is equal to 36; and if twice the greater be subtracted from 6 times the lesser, and the remainder divided by 8, the quotient will be 4. What are the numbers?

Let $x=$ the *greater* number,
$y=$ the *lesser* number;

$$\left.\begin{array}{r} \text{Then } 3x+\frac{y}{3}=36 \\ \frac{6y-2x}{8}=4 \end{array}\right\} \text{or, } \begin{array}{l} 9x+y=108 \\ 6y-2x=32; \end{array}$$

$$\text{Or, } y+9x=108 \ (A),$$
$$6y-2x=32 \ (B).$$

Multiply equation (A) by 6, then $6y+54x=648$
Subtract equation (B) . . . $6y-2x=32;$

$$\text{then } 56x=616;$$

$$\therefore x=\frac{616}{56}=11.$$

From equation A . . . . $y=108-9x=108-99=9.$

Qu. 9.

QUESTION 9.

There is a certain fraction, such, that if I add 3 to the numerator, its value will be $\frac{1}{3}$d; and if I subtract *one* from the denominator, its value will be $\frac{1}{5}$th. What is the fraction?

$$\left.\begin{array}{l}\text{Let } x=\text{its numerator}\\ \quad y=\text{denominator}\end{array}\right\}\text{then the fraction is } \frac{x}{y}.$$

$$\left.\begin{array}{l}\textit{Add } 3 \text{ to the numerator, then } \dfrac{x+3}{y}=\dfrac{1}{3}\\ \textit{Subtract} \text{ one from denom}^{\text{r}}\text{., and } \dfrac{x}{y-1}=\dfrac{1}{5}\end{array}\right\}\text{or, }\begin{array}{l}3x+9=y\\ 5x=y-1.\end{array}$$

By *transposition*, $y-3x=9$ (A),

$y-5x=1$ (B).

Subtract equation (B) from (A), and we have

$$2x=8;$$

$$\therefore\ x=\frac{8}{2}=4 \text{ the numerator.}$$

From equation (A) $y=9+3x=9+12=21$ the denominator.

Hence the fraction required is $\frac{4}{21}$.

QUESTION 10.

A and *B* have certain sums of money; says *A* to *B*, give me 15*l*. of your money, and I shall have 5 times as much as you will have left; says *B* to *A*, give me 5*l*. of your money, and I shall have exactly as much as you will have left. What sum of money had each?

$$\left.\begin{array}{l}\text{Let } x=A\text{'s money}\\ \quad y=B\text{'s} \ldots\end{array}\right\}\text{then } x+15=\begin{array}{l}\text{what } A \text{ would have after}\\ \text{receiving } 15l. \text{ from } B.\end{array}$$

$y-15=$ what *B* would have left.

Again, $y+5=$ what *B* would have after receiving 5*l*. from *A*.

$x-5=$ what *A* would have left.

$$\text{Hence, by the question, }\left.\begin{array}{l}x+15=5\times(y-15=)5y-75,\\ \text{and } y+5=x-5.\end{array}\right\}$$

$$\text{By transposition, }\left.\begin{array}{l}5y-x=90\ (A),\\ \text{and } y-x=-10\ (B).\end{array}\right\}$$

Set down equation (A) $5y-\ x=90$.
Multiply eq^n. (B) by 5, $5y-5x=-50$.

Subtract (B) from (A) $4x=140$;

$$\therefore\ x=\frac{140}{4}=35\ A\text{'s money.}$$

From equation (A) $5y=90+x=90+35=125$;

$$\therefore\ y=\frac{125}{5}=25\ B\text{'s money.}$$

QUESTION 11.

A person bought a certain number of sheep for 94*l*; having lost 7 of them, he sold $\frac{1}{4}$th of the remainder of them at *prime cost* for 20*l*. How many sheep had he at first?

Let x=number of sheep he had at first.

Then $\frac{94}{x}=\frac{\text{whole sum}}{\text{number of sheep}}$=what *each* sheep cost.

Now $x-7$ =number remaining when 7 were lost;

$\therefore\ \frac{x-7}{4}$=the number sold for 20*l*.

But the *number sold* × *price of each*=amount of sheep sold.

Hence, by *substitution*, $\frac{x-7}{4}\times\frac{94}{x}=20$,

$$\text{or } (x-7)\times 94\ =80x,$$
$$\text{i.e. } 94x-658=80x,$$
$$\text{or } 94x-80x=658,$$
$$\therefore\ 14x=658;\ \text{or } x=\frac{658}{14}=47.$$

QUESTION 12.

A and *B* have the *same* income; *A* is extravagant, and contracts an annual debt amounting to $\frac{1}{7}$th of it; *B* lives upon $\frac{4}{5}$ths of it: at the end of 10 years, *B* lends *A* money enough to pay off his debts, and has then 160*l*. to spare. What is their income?

Let x = their income.

Then $\frac{1}{7}$th of x, or $\frac{x}{7}$ = A's *annual* debt,

and $10\times\frac{x}{7}$ or $\frac{10x}{7}$ = A's debt contracted in 10 *years*.

As B lives upon $\frac{4}{5}$ths of his income, he saves annually $\frac{1}{5}$th of it;

hence, $\frac{x}{5}$ = B's *annual* saving,

and $10\times\frac{x}{5}$, or $\frac{10x}{5}$, or $2x$ = B's savings in 10 *years*.

But, by the question, B's savings = A's debt + 160;

$$\therefore \text{ by substitution, } 2x=\frac{10x}{7}+160,$$

$$\text{or } 14x=10x+1120,$$

$$\text{and } 4x=1120;\ \text{or } x=\frac{1120}{4}=280l.$$

QUESTION 13.

A person was desirous of relieving a certain number of beggars by giving them 2*s*. 6*d*. each, but found that he had not money enough in his pocket by 3 shillings; he then gave them 2 shillings each, and had four shillings to spare. What money had he in his pocket; and how many beggars did he relieve?

Let x = money in his pocket (in *shillings*).
y = number of beggars.

Then $2\frac{1}{2}\times y$, or $\frac{5y}{2}$ = N°. of *shill*^{s}. which would have [been given at 2*s*. 6*d*. each.

and $2\times y$, or $2y$ = ... at 2*s*. each.

Hence, by the question, $\frac{5y}{2}=x+3$ (A),

and $2y=x-4$ (B).

Subtract (B) from (A), then $\frac{y}{2}=7$, or $y=14$, the number of [beggars,

From eq^n. (B), $x=2y+4=28+4=32$ shillings in his pocket.

Qu. 14.

QUESTION 14.

A person passed $\frac{1}{6}$th of his age in childhood, $\frac{1}{12}$th in youth, $\frac{1}{7}$th $+5$ in matrimony; he had *then* a son whom he survived 4 years, and who reached only $\frac{1}{2}$ the age of his father. At what age did this person die?

Let x = age of the person at the time of his death.

Then $\frac{x}{6}$ = time spent in *childhood*.

$\frac{x}{12}$ = ... in *youth*.

$\frac{x}{7}+5$ = ... in *matrimony*.

$\therefore \frac{x}{6}+\frac{x}{12}+\frac{x}{7}+5$ = age of the person when the son [was born,

and $x-\frac{x}{6}-\frac{x}{12}-\frac{x}{7}-5$ = interval between birth of the son [and the old man's death;

$\therefore x-\frac{x}{6}-\frac{x}{12}-\frac{x}{7}-5-4$ = age of the son when he died.

But, by the question, the son died at $\frac{1}{2}$ the age of his father,

$$\text{Hence, } x-\frac{x}{6}-\frac{x}{12}-\frac{x}{7}-9=\frac{x}{2}.$$

$$\text{Mult}^y\text{. by 12, then } 12x-2x-x-\frac{12x}{7}-108=6x,$$

$$\text{or } 3x-\frac{12x}{7}=108,$$

$$\text{and } 21x-12x=756;$$

$$\therefore 9x=756; \text{ or } x=\frac{756}{9}=84.$$

QUESTION 15.

To find a number, such, that whether it is divided into *two* or *three* equal parts, the continued product of the parts shall be equal to the same quantity.

Let x=the number required.

Then $\frac{x}{2}\times\frac{x}{2}$=continued product, when the number [is divided into *two* parts,

and $\frac{x}{3}\times\frac{x}{3}\times\frac{x}{3}$=continued product, when the number [is divided into *three* parts.

Hence, by the question, $\left.\right\}\frac{x}{2}\times\frac{x}{2}=\frac{x}{3}\times\frac{x}{3}\times\frac{x}{3}$, or $\frac{x^2}{4}=\frac{x^3}{27}$;

$\therefore 27x^2=4x^3$.

Divide by x^2, then $27=4x$,

and $x=\frac{27}{4}=6\frac{3}{4}$, the number required.

QUESTION 16.

There is a certain number, consisting of two digits. The *sum* of those digits is 5; and if 9 be added to the number itself, the digits will be inverted. What is the number?

Let x=*left-hand* digit.
y=*right-hand* digit.

Then by Art. 63. $10x+y$=the number itself,
and $10y+x$=the number with digits *inverted*.

Hence, by the question, $x+y=5$ (*A*),
and $10x+y+9=10y+x$, or $9x-9y=-9$, or $x-y=-1$ (*B*).
Subtract (*B*) from (*A*), then $2y=6$, and $y=3$,
$x=5-y=5-3=2$;
$\therefore$ the number is $(10x+y)$ 23.

Add 9 to this number, and it becomes 32, which is the number with the *digits inverted*.

QU. 17. What two numbers are those whose difference is 10; and if 15 be added to their sum, the whole will be 43?

ANSWER, 9, and 19.

QU. 18. There are two numbers whose difference is 14; and if 9 times the lesser be subtracted from 6 times the greater, the remainder will be 33; What are the numbers?

ANSW. 17, and 31.

Qu. 19. What number is that, to which if I add 20, and from $\frac{2}{3}$ds of this sum I subtract 12, the remainder shall be 10?

ANSWER, 13.

Qu. 20. What number is that, of which if I add $\frac{1}{3}$d, $\frac{1}{4}$th, and $\frac{2}{7}$ths together, the sum shall be 73?

ANSW. 84.

Qu. 21. Two persons, *A* and *B*, lay out equal sums of money in trade; *A gains* 120*l.*, and *B loses* 80*l.*; and now *A*'s money is *treble* of *B*'s; What sum had each at first?

ANSW. 180*l.*

Qu. 22. What number is that whose $\frac{1}{3}$d part exceeds its $\frac{1}{5}$th by 72?

ANSW. 540.

Qu. 23. There are two numbers whose sum is 37; and if 3 times the lesser be subtracted from 4 times the greater, and this difference divided by 6, the quotient will be 6. What are the numbers?

ANSW. 21, and 16.

Qu. 24. There are two numbers whose sum is 49; and if $\frac{1}{7}$th of the lesser be subtracted from $\frac{1}{5}$th of the greater, the remainder will be 5. What are the numbers?

ANSW. 35, and 14.

Qu. 25. What two numbers are those, to one-third the sum of which if I add 13, the result shall be 17; and if from half their difference I subtract *one*, the remainder shall be *two*?

ANSW. 9, and 3.

Qu. 26. There is a certain fraction, such, that if I add *one* to its numerator, it becomes $\frac{1}{2}$; if 3 be added to the denominator, it becomes $\frac{1}{3}$. What is the fraction?

ANSW. $\frac{5}{12}$.

Qu. 27. A person has two horses, and a saddle worth 10*l.*; if the saddle be put on the *first* horse, his value becomes *double* that of the *second;* but if the saddle be put on the *second* horse, *his* value will not amount to that of the *first* horse by 13*l.* What is the value of each horse? Answ. 56 and 33.

Qu. 28. To divide the number 72 into three parts, so that $\frac{1}{2}$ the *first* part shall be equal to the *second*, and $\frac{3}{5}$ths of the *second* part equal to the third. Answ. 40, 20, and 12.

Qu. 29. A person after spending $\frac{1}{5}$th of his income *plus* 10*l.*, had then remaining $\frac{1}{2}$ of it *plus* 35*l.* Required his income. Answ. 150*l.*

Qu. 30. A gamester at *one sitting* lost $\frac{1}{5}$th of his money, and then won 10 shillings; at a *second* he lost $\frac{1}{3}$d of the remainder, and then won 3 shillings; after which he had 3 guineas left. What money had he at first? Answ. 5*l.*

Qu. 31. There are two numbers, such, that $\frac{1}{2}$ the greater added to $\frac{1}{3}$ the lesser is 13; and if $\frac{1}{2}$ the lesser be taken from $\frac{1}{3}$d the greater, the remainder is nothing. What are the numbers? Answ. 18 and 12.

Qu. 32. There is a certain number, to the sum of whose digits if you add 7, the result will be three times the left-hand digit; and if from the number itself you subtract 18, the digits will be *inverted.* What is the number? Answ. 53.

Qu. 33. Divide the number 90 into four such parts, that the first *increased* by 2, the second *diminished* by 2, the third *multiplied* by 2, and the fourth *divided* by 2, may all be equal to the same quantity. Answ. 18, 22, 10, 40.

Qu. 34. A merchant has two kinds of tea, one worth 9*s.* 6*d.* per pound, the other 13*s.* 6*d.* How many pounds of each must he take to form a chest of 104 lbs. which shall be worth 56*l.*?

Answ. 33 at 13*s.* 6*d.*
71 at 9*s.* 6*d.*

Qu. 35. A vessel containing 120 gallons is filled in 10 minutes by two spouts running *successively*; the one runs 14 gallons in a minute, the other 9 gallons in a minute. For what time has *each* spout run?

Answer, 14 gallon spout runs 6 minutes.
9 gallon spout . . . 4 minutes.

Qu. 36. In the composition of a certain quantity of gunpowder, $\frac{2}{3}$ds the whole + 10 was *nitre*; $\frac{1}{6}$th the whole − $4\frac{1}{2}$ was *sulphur*; and the charcoal was $\frac{1}{7}$th of the *nitre* − 2. How many pounds of gunpowder were there?

Answ. 69 pounds.

Qu. 37. To find three numbers, such, that the *first* with $\frac{1}{2}$ the sum of the *second* and *third* shall be 120; the *second* with $\frac{1}{5}$th the difference of the *third* and *first* shall be 70; and $\frac{1}{2}$ the sum of the three numbers shall be 95.

Answ. 50, 65, 75.

CHAP. V.

ON QUADRATIC EQUATIONS.

QUADRATIC Equations are divided into *pure* and *adfected*. *Pure* quadratic equations are those which contain only the *square* of the unknown quantity; such as $x^2=36$; $x^2+5=54$; $ax^2-b=c$; &c. *Adfected* quadratic equations are those which involve both the *square* and *simple power* of the unknown quantity, such as $x^2+4x=45$; $3x^2-2x=21$; $ax^2+2bx=c+d$; &c. &c.

XX.

On the Solution of Pure Quadratic Equations.

79. The Rule for the solution of pure quadratic equations is this, "Transpose the terms of the equation in such a manner, "that those which contain x^2 may be on one side of the "equation, and the *known quantities* on the other; divide "(if necessary) by the coefficient of x^2; then extract the square "root of each side of the equation, and it will give the value "of x."

Ex. 1.

Let $x^2+5=54$.

By transposition, $x^2=54-5=49$.

Extract the square root of both sides of the equation . . . . . . } then $x=\sqrt{49}=7$.

Ex. 2.

Ex. 2.

Let $3x^2-4=71$.

By transposition, $3x^2=71+4=75$.

Divide by 3, $x^2=\frac{75}{3}=25$.

Extract the square root, $x=\sqrt{25}=5$.

Ex. 3.

Let $5x^2-27=3x^2+215$.

By transposition, $5x^2-3x^2=215+27$,

or, $2x^2=242$;

$\therefore\ x^2=\frac{242}{2}=121$, and $x=11$.

Ex. 4.

Let $ax^2-b=c$;

then $ax^2=c+b$,

and $x^2=\frac{c+b}{a}$, or $x=\sqrt{\frac{c+b}{a}}$.

Ex. 5.

Let $ax^2-5c=bx^2-3c+d$.

Then $ax^2-bx^2=5c-3c+d$,

or $(a-b)x^2=2c+d$;

$\therefore\ x^2=\frac{2c+d}{a-b}$, and $x=\sqrt{\frac{2c+d}{a-b}}$.

Ex. 6. $5x^2-1=244$ . . . ANSWER, $x=7$.

Ex. 7. $9x^2+9=3x^2+63$ . . . . . $x=3$.

Ex. 8. $\frac{4x^2+5}{9}=45$ . . . . $x=10$.

Ex. 9. $bx^2+c+3=2bx^2+1$ . . . . . $x=\sqrt{\frac{c+2}{b}}$.

Ex. 10. $2ax^2+b-4=cx^2-5+d-ax^2$ . . $x=\sqrt{\frac{d-b-1}{3a-c}}$.

XXI.

On the Solution of Adfected Quadratic Equations.

80. The most general form under which an adfected quadratic equation can be exhibited is $ax^2+bx=c$; where a, b, c may be any quantities whatever, *positive* or *negative*, *integral* or *fractional*. Divide each side of this equation by a, then $x^2+\frac{b}{a}x=\frac{c}{a}$. Let $\frac{b}{a}=p$, $\frac{c}{a}=q$; then this equation is reduced to the form $x^2+px=q$, where p and q may be any quantities whatever, positive or negative, integral or fractional.

81. From the twofold form under which adfected quadratic equations may be expressed, there arise two Rules for their solution.

RULE I.

Let $x^2 \pm px=q$.

Add $\frac{p^2}{4}$ to each side of the equation, then $x^2 \pm px+\frac{p^2}{4}{}^{(a)}=\frac{p^2}{4}+q=\frac{p^2+4q}{4}$.

Extract the square root of each side of the equation, then $x\pm\frac{p}{2}=\frac{\pm\sqrt{p^2+4q}}{2}{}^{(b)}$

and $x=\frac{\pm\sqrt{p^2+4q}\mp p}{2}$.

(a) This quantity is the square of $x\pm\frac{p}{2}$, for

$$\begin{array}{r}
x\pm\frac{p}{2} \\
x\pm\frac{p}{2} \\
\hline
x^2\pm\frac{px}{2} \qquad \\
\pm\frac{px}{2}+\frac{p^2}{4} \\
\hline
x^2\pm px+\frac{p^2}{4} \\
\hline
\end{array}$$

(b) Since the square of $+a$ is $+a^2$, and of $-a$ is *also* $+a^2$, the *square root* of $+a^2$ may be either $+a$ or $-a$; hence the square root of p^2+4q may be expressed by $\pm\sqrt{p^2+4q}$.

Hence it appears, that "if to each side the equation there be added the *square of half the coefficient of the second term*, there will arise, on the left-hand side of the equation, a quantity which is the square of $x \pm \frac{p}{2}$; and by extracting the square root of each side of the resulting equation, we obtain a *simple* equation, from which the value of x may be determined."

RULE II.

Let $ax^2 \pm bx = c$.

Multiply each side of the equation by $4a$, } then $4a^2x^2 \pm 4abx = 4ac$.

Add b^2 to each side, we have $4a^2x^2 \pm 4abx + b^2$(c) $= 4ac + b^2$.

Extract the square-root as before, $2ax \pm b = \pm\sqrt{4ac+b^2}$

$$\therefore\ 2ax = \pm\sqrt{4ac+b^2} \mp b$$

$$\text{and } x = \frac{\pm\sqrt{4ac+b^2} \mp b}{2a}$$

From which we infer, that "if each side of the equation be multiplied by *four times the coefficient of* x^2, and to each side there be added *the square of the coefficient of* x, the quantity on the left-hand side of the equation will be the square of $2ax \pm b$. Extract the square root of each side of the equation, and there arises a *simple equation*, from which the value of x may be determined (d)."

If

(c) This quantity is the square of $2ax \pm b$, for

$$\begin{array}{l} 2ax \pm b \\ \underline{2ax \pm b} \\ 4a^2x^2 \pm 2abx \\ \qquad\quad\ \underline{\pm 2abx + b^2} \\ 4a^2x^2 \pm 4abx + b^2 \end{array}$$

(d) The principle of this Rule will be found in the *Bija Ganita*, a *Hindoo* Treatise on the Elements of Algebra. For a full account of that work, as translated by Mr. STRACHEY, see Dr. HUTTON'S *Tracts*, Vol. II. Tract 32.

If $a=1$, the equation is reduced to the form $x^2 \pm px = q$; in this case, therefore, the Rule may be applied, by "multiplying each side of the equation by 4, and adding the square "of the coefficient of x."

82. Either of these Rules may of course be applied to the solution of adfected quadratic equations; but it may be proper to observe, that in equations of the form $ax^2 \pm bx = c$ where a is a *small* number, and in those of the form $x^2 \pm px = q$ where p is an *odd* number, RULE II. will be found by far the most convenient.

83. From the form in which the value of x is exhibited in each of these Rules, it is evident that it will have *two* values; one corresponding to the sign +, and the other to the sign −, of the radical quantity. In the following examples, the *positive* values only of x are inserted at the end of the solution.

EXAMPLE I.

Let $x^2 + 8x = 65$.

By RULE I, add the square of 4 $\left(i.e.\ \frac{p^2}{4}\right)$ to each side of the equation, then . . . $x^2 + 8x + 16 = 65 + 16 = 81$.

Extract the square root of each side of the equation, then

$\left(x + \frac{p}{2}, \text{ or}\right)$ . . . . $x + 4 = \sqrt{81} = 9$,

and $x = 9 - 4 = 5$.

Ex. 2.

Let $x^2 - 4x = 45$.

By RULE I, add the square of 2, *i.e.* 4, then } . . . $x^2 - 4x + 4 = 45 + 4 = 49$.

Extract the square root, and $\left(x - \frac{p}{2}, \text{ or}\right)$ $x - 2 = \sqrt{49} = 7$,

and $x = 7 + 2 = 9$.

Ex. 3.

Ex. 3.

Let $3x^2+5x=42$.

By RULE II, multiply each side of the equation by $(4a)$ 12; then - - - - $\}$ $36x^2+60x=504$.

Add (b^2) 25 to each side of the equation, we have $\}$ $36x^2+60x+25=504+25=529$.

Extract the square root of each side the equation, which gives $(2ax+b$, or$)$ - - - - - - $6x+5=\sqrt{529}=23$

$$\therefore 6x=23-5=18$$

$$\text{and } x=\frac{18}{6}=3.$$

Ex. 4.

Let $7x^2\times 20x=32$,

$$\therefore x^2\times\frac{20x}{7}=\frac{32}{7}.$$

Complete the Square by RULE I,

$$\text{then } x^2\times\frac{20x}{7}+\frac{100}{49}=\frac{32}{7}+\frac{100}{49}=\frac{224}{49}+\frac{100}{49}=\frac{324}{49}.$$

$$\text{Hence } x\times\frac{10}{7}=\sqrt{\frac{324}{7}}=\frac{18}{7}$$

$$\text{and } x=\frac{18}{7}+\frac{10}{7}=\frac{28}{7}=4.$$

Ex. 5.

Let $x^2-15x=-54$.

By RULE II, multiply by 4 . . . $\}$ then $4x^2-60x=-216$.

Add (b^2) 225 to each side $\}$ and $4x^2-60x+225=225-216=9$.

Extract the square root $2x-15=\pm\sqrt{9}=\pm3$

$$\therefore 2x=15\pm3=18 \text{ or } 12$$

$$\text{and } x=\frac{18}{2} \text{ or } \frac{12}{2}=9 \text{ or } 6.$$

Ex. 6.

Let $4x^2-3x=85$.

By Rule II. multiply by 16, and add square of 3 to each side of the equation - $\Big\}$ $64x^2-48x+9=1360+9=1369$.

Extract the square root $\Big\}$ $8x-3=\sqrt{1369}=37$

$\therefore 8x=37+3=40$, or $x=\frac{40}{8}=5$.

Ex. 7.

Let $\frac{4x^2}{3}-11=\frac{x}{3}$.

Multiply by 3, then $4x^2-33=x$.

By transposition . . . $4x^2-x=33$.

Multiply by 16, and add 1 to each side of the equation (RULE II.) $\Big\}$ $64x^2-16x+1=528+1=529$.

Extract the square root, $8x-1=\sqrt{529}=23$

or $8x=23+1=24$,

and $x=\frac{24}{8}=3$.

Ex. 8.

Let $5x^2+4x=273$;

then $x^2+\frac{4x}{5}=\frac{273}{5}$

and by RULE I. $x^2+\frac{4x}{5}+\frac{4}{25}=\frac{273}{5}+\frac{4}{25}=\frac{1369}{25}$

$\therefore x+\frac{2}{5}=\sqrt{\frac{1369}{25}}=\frac{37}{5}$

and $x=\frac{37}{5}-\frac{2}{5}=\frac{35}{5}=7$.

Ex. 9.

Ex. 9.

$$\text{Let } \frac{7}{x+1}+\frac{2}{x}=5.$$

Mult. by $x+1$, then $7+\frac{2x+2}{x}=5x+5$.

. . . by x . . . . $7x+2x+2=5x^2+5x$.

By transposition, $5x^2-4x=2$.

By RULE II. $100x^2-80x+16=40+16=56$.

Extract the square root, $10x-4=\sqrt{56}$

and $10x=\sqrt{56}+4=7.48+4=11.48$

$$\therefore\ x=\frac{11.48}{10}=1.148.$$

Ex. 10.

Let $13x^2+2x=60$.

Divide by 13, $x^2+\frac{2x}{13}=\frac{60}{13}$.

By RULE I. add the square of $\frac{1}{13}$, $x^2+\frac{2x}{13}+\frac{1}{169}=\frac{60}{13}+\frac{1}{169}=\frac{780}{169}+\frac{1}{169}=\frac{781}{169}$.

Extract the square root $x+\frac{1}{13}=\frac{\sqrt{781}}{13}=\frac{27.94}{13}$.

$$\therefore\ x=\frac{27.94-1}{13}=\frac{26.94}{13}=2.07.$$

Ex. 11.

Let $2bx^2-cx=d$.

By RULE II. multiply by $8b$, and add c^2, $16b^2x^2-8bcx+c^2=8bd+c^2$.

Extract the square root $4bx-c=\sqrt{8bd+c^2}$

$\therefore\ 4bx=\sqrt{8bd+c^2}+c$.

or $x=\frac{\sqrt{8bd+c^2}+c}{4b}$.

Ex. 12. $x^2+12x=108$ . . . . $x=6.$

Ex. 13. $x^2-14x=51$ . . . . $x=17.$

Ex. 14. $x^2+6bx=c^2$ . . . . $x=\sqrt{c^2+9b^2}-3b.$

Ex. 15. $3x^2+2x=161$ . . . . $x=7.$

Ex. 16. $2x^2-5x=117$ . . . . $x=9.$

Ex. 17. $3x^2-2x=280$ . . . . $x=10.$

Ex. 18. $7x^2-20x=32$ . . . . $x=4.$

Ex. 19. $5x^2+4x=273$ . . . . . . $x=7.$

Ex. 20. $4x^2-7x=492$ . . . . $x=12.$

Ex. 21. $\frac{x^2}{6}-1=x+11$ . . . . $x=12.$

Ex. 22. $\frac{2x}{3}+\frac{1}{x}=\frac{7}{3}$ . . . . $x=3$ or $\frac{1}{2}.$

Ex. 23. $\frac{x^2}{3}-\frac{x}{2}=9$ . . . . $x=6.$

Ex. 24. $\frac{6}{x+1}+\frac{2}{x}=3$ . . . . $x=2.$

Ex. 25. $x^2-34=\frac{1}{3}x$ . . . . $x=6.$

Ex. 26. $\frac{x}{5}+\frac{5}{x}=5\frac{1}{5}$ . . . . $x=25$ or $1.$

Ex. 27. $\frac{x}{x+1}+\frac{x+1}{x}=\frac{13}{6}$ . . . . . $x=2.$

Ex. 28. $\frac{3x}{x+2}-\frac{x-1}{6}=x-9$ . . . . $x=10.$

Ex. 29. $x^2-6x+19=13$ . . . . $x=4.732$ or $1.268.$

Ex. 30. $5x^2+4x=25$ . . . . $x=1.871.$

Ex. 31. $4ax^2-bx=c$ . . . . $x=\frac{b\pm\sqrt{b^2+16ac}}{8a}.$

Ex. 32. $\frac{x}{a}+\frac{a}{x}=\frac{2}{a}$ . . . . $x=1\pm\sqrt{1-a^2}.$

XXII.

On the Solution of Questions producing Quadratic Equations.

84. In the solution of Questions which involve quadratic equations, sometimes *both*, and sometimes only *one* of the values of the unknown quantity, will answer the conditions required. This is a circumstance which may always be very readily determined by the nature of the question itself.

QUESTION 1.

To divide the number 56 into two such parts, that their product shall be 640.

Let $x =$ *one* part,

then $56 - x =$ the *other* part,

and $x(56-x) =$ *product* of the two parts.

Hence, by the question, $x(56-x) = 640$,

or $56x - x^2 = 640$.

By transposition, $x^2 - 56x = -640$.

By completing the square, (RULE 1.) $x^2 - 56x + 784 = 784 - 640 = 144$;

$\therefore\ x - 28 = \pm\sqrt{144} = \pm 12$,

and $x = 28 \pm 12 = 40$ or 16.

In this case it appears that the *two* values of the unknown quantity are the *two* parts into which the given number was required to be divided.

QUESTION 2.

There are two numbers whose difference is 7, and half their product *plus* 30 is equal to the square of the *lesser* number. What are the numbers ?

Let $x =$ the *lesser* number,

then $x + 7 =$ the *greater* number,

and $\frac{x \times (x+7)}{2} + 30 =$ half their product *plus* 30.

Hence, by the question, $\frac{x\times(x+7)}{2}+30=x^2$ (square of *lesser*),

$$\text{or } \frac{x^2+7x}{2}+30=x^2.$$

Multiply by 2 . . . . $x^2+7x+60=2x^2.$

By transposition . . . . . $x^2-7x=60.$

Multiply by 4, and add 49 (RULE II.) $4x^2-28x+49=240+49=289,$

$\therefore\ 2x-7=\sqrt{289}=17$

$2x=17+7=24$, or $x=12$ *lesser* number;

hence $x+7=12+7=19$ *greater* number.

QUESTION 3.

To divide the number 30 into two such parts, that their product may be equal to *eight* times their difference.

Let $x=$ the *lesser* part,

then $30-x=$ the *greater* part,

and $30-x-x$ or $30-2x=$ their *difference.*

Hence, by the question, $x\times(30-x)=8\times(30-2x)$,

or $30x-x^2=240-16x.$

By transposition, $x^2-46x=-240.$

Complete the square, (RULE I.) $x^2-46x+529=529-240=289;$

$\therefore\ x-23=\pm\sqrt{289}=\pm17,$

and $x=23\pm17=40$ or $6=$ *lesser* part;

$30-x=30-6=24=$ *greater* part.

In this case, the solution of the equation gives 40 and 6 for the *lesser* part. Now as 40 cannot possibly be a *part* of 30, we take 6 for the *lesser* part, which gives 24 for the *greater* part; and the two numbers, 24 and 6, answer the conditions required.

QUESTION 4.

A person bought cloth for 33*l.* 15*s.* which he sold again at 2*l.* 8*s.* per piece, and gained by the bargain as much as one piece cost him; Required the number of pieces.

Let x=the number of pieces.

Then $\frac{675}{x}$=number of *shillings* each piece *cost*,

and $48x$=number of *shillings* he *sold* the whole for;

$\therefore$ $48x-675$=what he gained by the bargain.

Hence, by the question, $48x-675=\frac{675}{x}$.

By transposition and division, $\left.\right\}$ $x^2-\frac{225}{16}x=\frac{225}{16}$.

Complete the square, (RULE I.) $\left.\right\}$ $x^2-\frac{225}{16}x+\frac{50625}{1024}=\frac{225}{16}+\frac{50625}{1024}=\frac{65025}{1024}$;

$$\therefore\ x-\frac{225}{32}=\sqrt{\frac{65025}{1024}}=\frac{255}{32},$$

$$\text{and } x=\frac{255+225}{32}=15.$$

QUESTION 5.

A and *B* set off at the *same time* to a place at the distance of 150 miles. *A* travels 3 miles an hour *faster* than *B*, and arrives at his journey's end 8 hours and 20 minutes *before* him. At what rate did each person travel per hour?

Let x=rate per hour at which *B* travels.

Then $x+3$= . . . . *A* . . . .

And $\frac{150}{x}$=number of hours for which *B* travels.

$\frac{150}{x+3}$= . . . . *A* . . . .

But *A* is 8 hours 20 minutes ($8\frac{1}{3}$ hours) *sooner* at his journey's end than *B*;

$$\text{Hence } \frac{150}{x+3}+8\tfrac{1}{3}=\frac{150}{x},$$

$$\text{or } \frac{150}{x+3}+\frac{25}{3}=\frac{150}{x}.$$

By reduction, $x^2+3x=54$.

Complete the square, $x^2+3x+\frac{9}{4}=54+\frac{9}{4}=\frac{225}{4}$ (Rule I.);

$$\therefore\ x+\frac{3}{2}=\sqrt{\frac{225}{4}}=\frac{15}{2};$$

and $x=\frac{15-3}{2}=6$ miles an hour for *B*

$x+3=9$ . . . . for *A*.

Question 6.

Some bees had alighted upon a tree; at one flight the square root of half of them went away; at another $\frac{8}{9}$ths of them; two bees then remained. How many alighted on the tree?[a]

Let $2x^2=$ the N° of bees;

then $x+\frac{16x^2}{9}+2=2x^2$,

or $9x+16x^2+18=18x^2$;

$\therefore\ 18x^2-16x^2-9x=18$,

or $2x^2-9x=18$.

(Rule II.) Multiply by 8,

$16x^2-72x=144$.

Add 81; then $16x^2-72x+81=225$,

or $4x-9=15$;

$\therefore\ 4x=15+9=24$,

and $x=\frac{24}{4}=6$; $\therefore\ 2x^2=72$ N° of bees.

Qu. 7. To divide the number 33 into two such parts, that their product shall be 162.

Answer, 27 and 6.

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[a] This question is taken from Mr. Strachey's Translation of the *Bija Ganita*; and the several steps of the operation will, upon comparison, be found to accord with the *Hindoo* method of solution, as it stands in that Translation, page 62.

Qu. 8. What two numbers are those whose sum is 29, and product 100? Answer, 25 and 4.

Qu. 9. The difference of two numbers is 5, and $\frac{1}{4}$th part of their product is 26. What are the numbers?

Answ. 13 and 8.

Qu. 10. The difference of two numbers is 6; and if 47 be added to *twice the square of the lesser*, it will be equal to the *square of the greater*. What are the numbers?

Answ. 17 and 11.

Qu. 11. There are two numbers whose sum is 30; and $\frac{1}{3}$d of their product *plus* 18 is equal to the square of the *lesser* number. What are the numbers?

Answ. 21 and 9.

Qu. 12. There are two numbers whose product is 120. If 2 be added to the lesser, and 3 subtracted from the greater, the product of the sum and remainder will also be 120. What are the numbers? Answ. 15 and 8.

Qu. 13. *A* and *B* distribute 1200*l*. each among a certain number of persons: *A* relieves 40 persons more than *B*, and *B* gives 5*l*. apiece to each person *more* than *A*. How many persons were relieved by *A* and *B* respectively?

Answ. 120 by *A*, 80 by *B*.

Qu. 14. A person bought a certain number of sheep for 120*l*. If there had been 8 more, each sheep would have cost him 10 shillings less. How many sheep were there?

Answ. 40.

Qu. 15. A person bought a certain number of sheep for 57*l*. Having lost 8 of them, and sold the remainder at 8 shillings a head profit, he is no loser by the bargain. How many sheep did he buy? Answ. 38.

QU. 16. *A* and *B* set off at the same time to a place at the distance of 300 miles. *A* travels at the rate of one mile an hour faster than *B*, and arrives at his journey's end 10 hours before him. At what rate did each person travel per *hour?*

ANSW. *A* travelled 6 miles per hour.
B . . . . . 5 . . . .

XXIII.

On Quadratic Equations having Impossible Roots.

85. In the solution of the adfected quadratic equation, $x^2+px=q$ (Art. 81.) the two values of x were shewn to be equal to $\frac{\pm\sqrt{p^2+4q}-p}{2}$. If q be a *negative* quantity, and p^2 less than $4q$, then the quantity p^2-4q is negative, and consequently the quantity $\pm\sqrt{p^2-4q}$ comes under the description of the radical quantities mentioned in Art. 58. In this case, the two roots, or values of x, are said to be *impossible*.

Ex. 1.

Let $x^2+8x+31-0$, or $x^2+8x=-31$,

Complete the square, (RULE I.)

$$\text{then } x^2+8x+16=-31+16=-15,$$
$$\text{and } x+4=\pm\sqrt{-15},$$
$$\text{or } x=-4\pm\sqrt{-15}.$$

Ex. 2.

Let $x^2-12x+50=0$, or $x^2-12x=-50$.

Complete the square, (RULE I.)

$$x^2-12x+36=-50+36=-14,$$
$$\text{and } x-6=\pm\sqrt{-14};$$
$$\therefore x=6\pm\sqrt{-14}.$$

Ex. 3.

To find the number 16 into two such parts, that their product shall be equal to 70.

Let x = one part,

then $16-x$ = the other part.

Hence $x(16-x)$ or $16x-x^2=70$.

Transpose, and $x^2-16x=-70$.

Complete the square,

$$x^2-16x+64=-70+64=-6,$$

$$\therefore x-8=\pm\sqrt{-6}, \text{ or } x=8\pm\sqrt{-6}.^{(a)}$$

Ex. 4. $2x^2+15=3x$ . . . . . $x=\dfrac{3\pm\sqrt{-111}}{4}$

Ex. 5. $3x-\frac{1}{4}x^2=10$ . . . . . $x=6\pm\sqrt{-4}$.

Ex. 6. To divide the number 20 into two such parts, that their product shall be 105 . . . . $x=10\pm\sqrt{-5}$.

XXIV.

On the Solution of Quadratic Equations of the form $x^{2n}+px^n=q$.

86. Let $y=x^n$, then (by Case III. Art. 67.) $y^2=x^{2n}$; and substituting these values for x^{2n} and x^n in the equation $x^{2n}+px^n=q$, it is transformed into $y^2+py=q$, where the value of y may be determined by the foregoing Rules. Having the value of y, the value of x may be found; for $x^n=y$, $\therefore x=\sqrt[n]{y}$. We are thus enabled to solve equations in which the unknown quantity is found only in *two* terms, (and where the index of the highest power is *double* the index of the lowest), like common quadratics.

Ex. 1.

(a) It is very well known that the *greatest* product which can arise from the multiplication of the two parts into which any given number may be divided, is when these two parts are *equal*: the greatest product therefore, which could arise from the division of the number 16 into two parts, is when each of them is 8; hence, in requiring "to divide the number 16 into two such parts that their product should be 70," the solution of the question is *impossible*.

Ex. 1.

Let $x^4-6x^2=27$.

$$\left.\begin{array}{l}\text{If } x^2=y,\\ \text{then } x^4=y^2\end{array}\right\} \therefore y^2-6y=27.$$

By Rule I. $y^2-6y+9=27+9=36$,

and $y-3=\sqrt{36}=6$,

or $y=6+3=9$.

But since $x^2=y$, $x=\sqrt{y}$, $\therefore x=\sqrt{9}=3$.

Ex. 2.

Let $x^6-2x^3=48$.

These equations are often solved by the common Rules, without the formality of substitution; thus,

Complete the square, (Rule I.) $x^6-2x^3+1=48+1=49$.

Extract the roots $x^3-1=\sqrt{49}=7$,

$\therefore x^3=7+1=8$

$x=\sqrt[3]{8}=2$.

Ex. 3.

Let $2x-7\sqrt{x}=99$.

$$\left.\begin{array}{l}\text{Put } y^2=x\\ \text{then } y=\sqrt{x}\end{array}\right\} \therefore 2y^2-7y=99.$$

By Rule II. $16y^2-56y+49=792+49=841$,

and $4y-7=\sqrt{841}=29$,

or $4y=29+7=36$, and $y=9$;

$\therefore x=y^2=81$.

Ex. 4.

To resolve the number a into two such factors, that the sum of their nth powers shall be equal to b.

Let x = one factor (a),

then $\frac{a}{x}$ = the other factor.

(a) By *factors* are here meant the two numbers which being multiplied together shall generate the given number; if therefore x = one factor, $\frac{a}{x}$ must be the other factor, for $x\times\frac{a}{x}=a$.

Hence $x^n+\frac{a^n}{x^n}=b$,

or $x^{2n}+a^n=bx^n$;

$\therefore\ x^{2n}-bx^n=-a^n$.

By Rule II.

$4x^{2n}-4bx^n+b^2=b^2-4a^n$,

and $2x^n-b=\sqrt{b^2-4a^n}$,

or $2x^n=b+\sqrt{b^2-4a^n}$, and $x^n=\frac{b+\sqrt{b^2-4a^n}}{2}$;

$$\therefore x=\sqrt[n]{\frac{b+\sqrt{b^2-4a^n}}{2}}.$$

Ex. 5. $x^4+4x^2=12$ . . . . $x=\sqrt{2}$.

Ex. 6. $x^6-8x^3=513$ . . . . $x=3$.

Ex. 7. $2x^4-x^2=496$ . . . . $x=4$.

Ex. 8. To resolve the number 18 into two such factors, that the sum of their *cubes* shall be 243 (See Ex. 4.)

Answer, 6 and 3.

XXV.

On the Solution of Quadratic Equations containing Two unknown Quantities.

The solution of equations with *two* unknown quantities, in which one or both these quantities are found in a quadratic form, can only, in *particular cases* (b), be effected by means of the preceding Rules. Of these cases the two following are very well known.

(b) The most complete form under which quadratic equations containing two unknown quantities could be expressed, is this,

$ax^2+by^2+cxy+dx+ey=m$

$a'x^2+b'y^2+c'xy+d'x+e'y=m'$; but the general solution of these equations can only be affected by means of equations of higher dimensions than quadratics.

CASE I.

87. "When one of the equations by which the values of "the unknown quantities are to be determined, is a *simple* "equation;" in which case, the Rule is, "to find a value of one "of the unknown quantities from that simple equation, and "then substitute for it the value so found, in the other equa-"tion; the resulting equation will be a quadratic, which may "be solved by the ordinary Rules." Thus,

Let $\left.\begin{array}{l} ax+by=c \\ a'x^2+b'xy+c'y^2=d \end{array}\right\}$ be the two equations, in which the values of x and y are to be determined.

From the 1st equation $x=\frac{c-by}{a}$.

Substitute this for x in the 2d equation, $\Big\}$ then $a'\left(\frac{c-by}{a}\right)^2+b'\left(\frac{cy-by^2}{a}\right)+c'y^2=d$,

which reduced is $(a'b-bb'+ac')y^2+(b'c-2a'bc)y=ad-a'c^2$, a common quadratic equation, from which the value of y may be found.

Ex. 1.

Let $\left.\begin{array}{l} x+2y=7, \\ \text{and } x^2+3xy-y^2=23 \end{array}\right\}$ to find the values of x and y.

From 1st equation, $x=7-2y$, $\therefore x^2=49-28y+4y^2$;
Substitute these values for x and x^2 in the 2d equation,
then $49-28y+4y^2+21y-6y^2-y^2=23$,
or $3y^2+7y=49-23=26$.

By RULE II. $36y^2+84y+49=312+49=361$,
$\therefore 6y+7=19$
$6y=19-7=12$, or $y=2$
$x=7-2y=7-4=3$.

Ex. 2.

Ex. 2.

$$\left.\begin{array}{r}\text{Let } \dfrac{2x+y}{3}=\ \ 9\\ \text{and } 3xy=210\end{array}\right\}\text{ to find the values of } x \text{ and } y.$$

From 1st equation, $2x+y=27$;

$$\therefore\ 2x=27-y,$$

$$\text{and } x=\frac{27-y}{2}.$$

$$\text{Hence, } 3xy=3\times\frac{27-y}{2}\times y=210,$$

$$\text{or } 3\times(27-y)\times y=420$$

$$81y-3y^2=420$$

$$27y-\ y^2=140;$$

$$\text{or } y^2-27y=-140.$$

By RULE II, $4y^2-108y+729=729-560=169$;

$$\therefore\ 2y-27=13,\text{ or } y=\frac{27+13}{2}=20,$$

$$\text{and } x=\frac{27-20}{2}=3\tfrac{1}{2}.$$

Ex. 3.

There is a certain number consisting of two digits. The left-hand digit is equal to 3 times the right-hand digit; and if 12 be subtracted from the number itself, the remainder will be equal to the square of the left-hand digit. What is the number?

Let x be the left-hand digit, and y the other; } then, by Art. 63, $10x+y$ is the number.

$$\left.\begin{array}{r}\text{Hence, } x=3y\\ \text{and } 10x+y-12=x^2\end{array}\right\}\text{ by the question;}$$

$\therefore$ by substitution } $30y+y-12=9y^2$, (for $10x=30y$, and $x^2=9y^2$);

$$9y^2-31y=-12;$$

$$\therefore\ y^2-\frac{31}{9}y=-\frac{12}{9},$$

By RULE I, $y^2-\frac{31}{9}y+\frac{961}{324}=\frac{961}{324}-\frac{12}{9}=\frac{961-432}{324}=\frac{529}{324}$.

$$\text{Hence, } y-\frac{31}{18}=\frac{23}{18}\text{; or } y=\frac{54}{18}=3,$$

$$x=3y=9;$$

and consequently the number is 93.

Ex. 4. Let $\left.\begin{array}{l}2x-3y=1\\ 2x^2+xy-5y^2=20\end{array}\right\}$ to find the values of x and y.

ANSWER, $x=5$, $y=3$.

Ex. 5. There are two numbers, such, that if the lesser be taken from three times the greater, the remainder will be 35; and if four times the greater be divided by three times the lesser *plus* one, the quotient will be equal to the lesser number. What are the numbers? ANSW. 13 and 4.

Ex. 6. What number is that, the *sum* of whose digits is 15, and if 31 be added to their *product*, the digits will be inverted? ANSW. 78.

CASE II.

88. When x^2, y^2, or xy, is found in every term of the two equations, they assume the form of

$$ax^2+bxy+cy^2=d,$$

$$a'x^2+b'xy+c'y^2=d'\text{; and their solution}$$

may be effected in the following manner.

Assume $x=vy$, then $x^2=v^2y^2$; substitute these values for x^2 and x in both equations, then we have

$$av^2y^2+bvy^2+cy^2=d,\text{ or } y^2=\frac{d}{av^2+bv+c}\ (A)$$

$$a'v^2y^2+b'vy^2+c'y^2=d',\text{ or } y^2=\frac{d'}{a'v^2+b'v+c'}\ (B).$$

$$\text{Hence } \frac{d}{av^2+bv+c}=\frac{d'}{a'v^2+b'v+c'}$$

or $(a'd-ad')v^2+(b'd-bd')v=cd'-c'd$; which is a quadratic equation, from whence the value of v may be determined. Having the value of v, the value of y may be found from either of the equations (A) or (B); and then the value of x, from the equation $x=vy$.

Ex. 1.

$$\text{Let } 2x^2+3xy+y^2=20$$
$$5x^2+4y^2=41;$$

Assume $x=vy$, then $2v^2y^2+3vy^2+y^2=20$, or $y^2=\frac{20}{2v^2+3v+1}$,

and $5v^2y^2+4y^2=41$, or $y^2=\frac{41}{5v^2+4}$;

$$\text{Hence } \frac{20}{2v^2+3v+1}=\frac{41}{5v^2+4},$$

which reduced is, $6v^2-41v=-13$;

$$\therefore\ v^2-\frac{41v}{6}=-\frac{13}{6}.$$

By Rule I, $v^2-\frac{41v}{6}+\frac{1681}{144}=\frac{1369}{144}$;

$$\therefore\ v-\frac{41}{12}=\frac{\pm 37}{12};\ \text{or } v=\frac{41\pm 37}{12}=\frac{13}{2}\ \text{or}\ \frac{1}{3}.$$

Let $v=\frac{1}{3}$, then $y^2=\frac{41}{5v^2+4}=\frac{41}{\frac{5}{9}+4}=\frac{369}{41}=9$, or $y=3$,

$$x=vy=\tfrac{1}{3}\times 3=1.$$

Ex. 2.

What two numbers are those, whose sum multiplied by the greater is 77? and whose difference multiplied by the lesser is equal to 12?

Let $x=$greater number,
$y=$lesser.

Then $(x+y)\times x=x^2+xy=77$,
and $(x-y)\times y=xy-y^2=12$.

Assume $x=vy$;

$$\left.\begin{array}{r}\text{Then } v^2y^2+vy^2=77,\\ \text{and } vy^2-y^2=12\end{array}\right\}\begin{array}{l}\text{or } y^2=\frac{77}{v^2+v};\\ \text{or } y^2=\frac{12}{v-1}.\end{array}$$

Hence,

$$\text{Hence, } \frac{12}{v-1}=\frac{77}{v^2+v},$$

$$\text{or } 12v^2+12v=77v-77;$$

$$\text{which gives } v^2-\frac{65}{12}v=-\frac{77}{12}$$

$$\text{and } v^2-\frac{65}{12}v+\frac{4225}{576}=\frac{529}{576};$$

$$\therefore\ v=\frac{65\pm23}{24}=\frac{88 \text{ or } 42}{24}=\frac{11}{3} \text{ or } \frac{7}{4}.$$

Either value of v will answer the conditions of the question; but take $v=\frac{7}{4}$; then $y^2=\frac{12}{v-1}=\frac{12}{\frac{7}{4}-1}=\frac{48}{7-4}=\frac{48}{3}=16$,

$$\text{and } y=4,$$

$$x=vy=\frac{7}{4}\times4=7.$$

Hence, the numbers are 4 and 7.

Ex. 3. Find two numbers, such, that the square of the greater *minus* the square of the less may be 56; and the square of the lesser *plus* $\frac{1}{3}$d their product may be 40.

ANSWER, 9 and 5.

Ex. 4. There are two numbers, such, that 3 times the square of the greater *plus* twice the square of the less is 110; and half their product *plus* the square of the lesser is 4. What are the numbers(a)? ANSW. 6 and 1.

XXVI.

On the Solution of certain Equations, in which the Two unknown Quantities (x and y) are similarly involved.

89. Let x and y be any two numbers, of which x is the greater, and y the lesser; let $x+y=2s$, $x-y=2z$; then, by Art.28, $x=s+z$, and $y=s-z$. Now let $x^2+y^2=a$, $x^3+y^3=b$, $x^4+y^4=c$, and $x^5+y^5=d$; then the values of x and y may be

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(a) For a great variety of questions relating to quadratic equations which contain two unknown quantities, see BLAND's *Algebraical Problems*, 1812.

found in terms of the known quantities s, a, b, c, d, in the following manner.

I. $x^2=(s+z)^2=s^2+2sz+z^2$

$y^2=(s-z)^2=s^2-2sz+z^2$;

∴ by addition,

$$x^2+y^2(a)=2s^2+2z^2, \text{ and } z^2=\frac{a-2s^2}{2} \text{ or } z=\sqrt{\frac{a-2s^2}{2}}.$$

Hence $x=s+\sqrt{\dfrac{a-2s^2}{2}}$, and $y=s-\sqrt{\dfrac{a-2s^2}{2}}$.

II. $x^3=(s+z)^3=s^3+3s^2z+3sz^2+z^3$

$y^3=(s-z)^3=s^3-3s^2z+3sz^2-z^3$;

$$\therefore x^3+y^3(b)=2s^3+6sz^2; \text{ and } z^2=\frac{b-2s^3}{6s}, \text{ or } z=\sqrt{\frac{b-2s^3}{6s}}.$$

Hence $x=s+\sqrt{\dfrac{b-2s^3}{6s}}$, and $y=s-\sqrt{\dfrac{b-2s^3}{6s}}$.

III. $x^4=(s+z)^4=s^4+4s^3z+6s^2z^2+4sz^3+z^4$,

$y^4=(s-z)^4=s^4-4s^3z+6s^2z^2-4sz^3+z^4$;

∴ $x^4+y^4(c)=2s^4+12s^2z^2+2z^4$ is a quadratic equation from which the value of z may be found.

IV. $x^5=(s+z)^5=s^5+5s^4z+10s^3z^2+10s^2z^3+5sz^4+z^5$,

$y^5=(s-z)^5=s^5-5s^4z+10s^3z^2-10s^2z^3+5sz^4-z^5$;

∴ $x^5+y^5(d)=2s^5+20s^3z^2+10sz^4$ is a quadratic equation from which the value of z, may be found(b).

(b) In reviewing these operations, it may be observed, that those terms where the index of z is an *odd* number destroy each other in the successive series; hence if the operations had been continued to x^6+y^6 and x^7+y^7, the resulting equations would have been equations of *six* dimensions in a *cubic* form; if they had been carried on to x^8+y^8 and x^9+y^9, the resulting equations would have been equations of *eight* dimensions in a *biquadratic* form. Hence the Problem of "Given the sum of two numbers, and the sum of their nth powers, to find the numbers themselves," may be solved as far as the 9th power, by means either of *quadratic*, *cubic*, or *biquadratic* equations.

90. Let $x+y=2s$ and $x-y=2z$ as before, and let $\frac{x}{y}+\frac{y}{x}=a'$; $\frac{x^2}{y}+\frac{y^2}{x}=b'$; $\frac{x^3}{y}+\frac{y^3}{x}=c'$; and $\frac{x^4}{y}+\frac{y^4}{x}=d'$; then, by means of the equations in the preceding article (89), the values of x and y may be found in terms of the known quantities, s, a', b', c', d'.

I. $\frac{x}{y}+\frac{y}{x}=a'$, $\therefore x^2+y^2=a'xy=a'(s+z)(s-z)=a''(s^2-z^2)$.

But by Case I. (89). $x^2+y^2=2s^2+2z^2$;

Hence $a's^2-a'z^2=2s^2+2z^2$,

and $z^2=\frac{(a'-2)s^2}{a'+2}$ or $z=\sqrt{\frac{(a'-2)s^2}{a'+2}}$;

$\therefore x=s+\sqrt{\frac{(a'-2)s^2}{a'+2}}$, and $y=s-\sqrt{\frac{(a'-2)s^2}{a'+2}}$.

II. $\frac{x^2}{y}+\frac{y^2}{x}=b'$, $\therefore x^3+y^3=b'xy=b'(s^2-z^2)$.

By Case II. (89). $x^3+y^3=2s^3+6sz^2$;

$\therefore b'(s^2-z^2)=2s^3+6sz^2$,

and $z^2=\frac{(b'-2s)s^2}{b'+6s}$, or $z=\sqrt{\frac{(b'-2s)s^2}{b'+6s}}$.

Hence $x=s+\sqrt{\frac{(b-2s)s^2}{b'+6s}}$, and $y=s-\sqrt{\frac{(b'-2s)s^2}{b'+6s}}$.

III. $\frac{x^3}{y}+\frac{y^3}{x}=c'$, $\therefore x^4+y^4=c'xy=c'(s^2-z^2)$.

By Case III. (89). $x^4+y^4=2s^4+12s^2z^2+2z^4$;

Hence $c'(s^2-z^2)=2s^4+12s^2z^2+2z^4$ is a quadratic equation, by which the value of z may be found.

IV. $\frac{x^4}{y}+\frac{y^4}{x}=d$, $\therefore x^5+y^5=dxy=d(s^2-z^2)$.

By CASE IV. (89). $x^5+y^5=2s^5+20s^3z^2+10sz^4$; and by equating these two values of x^5+y^5, there arises a quadratic equation by which the value of z may be determined.

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91. Let $x+y=s$, and $xy=p$; then the sums of the several powers of x and y may be found in terms of the known quantities p and s, in the following manner.

$$\text{I.}\quad x^2+2xy+y^2=s^2;$$
$$\therefore x^2+y^2=s^2-2xy=s^2-2p.$$

$$\text{II.}\quad (x^2+y^2)(x+y)=(s^2-2p)s,$$
$$\text{or } x^3+y^3+xy(x+y)=s^3-2ps,$$
$$\text{i.e. } x^3+y^3+ps=s^3-2ps;$$
$$\therefore x^3+y^3=s^3-3ps.$$

$$\text{III.}\quad (x^3+y^3)(x+y)=(s^3-3ps)s,$$
$$\text{or } x^4+y^4+xy(x^2+y^2)=s^4-3ps^2,$$
$$\text{i.e. } x^4+y^4+p(s^2-2p)=s^4-3ps^2;$$
$$\therefore x^4+y^4=s^4-4ps^2+2p^2.$$

$$\text{IV.}\quad (x^4+y^4)(x+y)=(s^4-4ps^2+2p^2)s,$$
$$\text{or } x^5+y^5+xy(x^3+y^3)=s^5-4ps^3+2p^2s,$$
$$\text{i.e. } x^5+y^5+p(s^3-3ps)=s^5-4ps^3+2p^2s;$$
$$\therefore x^5+y^5=s^5-5ps^3+5p^2s.$$

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or in general $x^n+y^n=s^n-nps^{n-2}+n\frac{(n-3)}{2}p^2s^{n-4}-\&c.$

Ex. 1.

The sum of two numbers is 6, and the sum of their fifth powers is 1056; What are the numbers?

This Example belongs to CASE IV. Art. 89, where $s=3$, and $d=1056$.

The equation to find the value of z is

$$2s^5+20s^3z^2+10sz^4=d,$$

$$\text{or } 486+540z^2+30z^4=1056;$$

Divide by 6, $81+90z^2+5x^4=176;$

$$\therefore z^4+18z^2=19.$$

By Rule I. $z^4+18z^2+81=100,$

$$\text{or } z^2+9=10;\ \therefore z^2=1, \text{ and } z=1.$$

$$\text{Hence } x=s+z=3+1=4,$$

$$y=s-z=3-1=2.$$

Ex. 2.

There are two numbers whose sum is 18, and the square of the greater divided by the lesser *plus* the square of the lesser divided by the greater is 27; What are the numbers?

In Case II. (90). $s=9$, and $b'=27$; hence $z=\sqrt{\frac{(b'-2s)s^2}{b'+6s}}=\sqrt{\frac{9\times81}{27+54}}=\sqrt{\frac{9\times81}{81}}=\sqrt{9}=3;\ \therefore x=s+z=9+3=12$, and $y=s-z=9-3=6$; and the two numbers are 12 and 6.

Ex. 3.

The sum of two numbers is 5 (s), and their product 6 (p); What is the sum of their 4th powers?

By Case III. (Art. 91): $x^4+y^4=s^4-4ps^2+2p^2=625-600+72=25+72=97$.

CHAP. VI.

ON RATIOS, PROPORTION, AND VARIABLE QUANTITIES.

XXVII.

Definitions.

92. BY RATIO is meant the relation which one quantity bears to another, with respect to magnitude. It is evident that this relation can exist only between quantities of a similar kind; thus, a *number* must be compared with a number; a *line* with a line; &c. &c.; and it would be absurd to compare a certain number of *feet* with a certain number of *pounds*; &c. &c.

93. There are two ways in which the magnitude of quantities may be compared. In the first place, they may be compared with regard to their *difference*; and then the question asked, is, "How much one quantity is greater or less than another." The relation which quantities bear to each other in this respect, is called their *Arithmetical* ratio. The other way in which they may be compared, is, by inquiring "How often one quantity is contained in the other." This relation between quantities is called their *Geometrical* Ratio. The term *ratio*, when simply applied, is generally understood in the latter sense; and it is in this sense that the word will be made use of in the present Chapter.

94. In considering how often one quantity is contained in another, the natural process is to *divide* the one by the other. Thus, in comparing the number 12 with the numbers 4 and 3,

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we know that 4 is contained in 12 *three* times, and that 3 is contained in the same number *four* times; from which we infer the ratio of 12 : 3[(a)] is *greater* than the ratio of 12 to 4, the *magnitude* of a ratio being measured by the *number of times* one quantity is contained in another. For the same reason, the ratio of 11 : 7 is said to be *less* than the ratio of 11 : 5. When a ratio is thus expressed, the first term of it is called the *antecedent*, the last term the *consequent*, of that ratio.

95. From this mode of estimating the magnitude of a ratio, it appears that when the consequent of a ratio is not an *aliquot part* of the antecedent, the value of the ratio must be expressed by a *fraction*, whose *numerator* is the antecedent, and *denominator* the consequent of that ratio. Thus the magnitude of the ratio of 15 : 7 is expressed by the fraction $\frac{15}{7}$, and of the ratio 4 : 13 by the fraction $\frac{4}{13}$. When the antecedent of a ratio is greater than the consequent, it is called a ratio of *greater inequality*; when the antecedent is less than the consequent, a ratio of *lesser inequality*; and if the two terms of a ratio be the *same*, then it is said to be a ratio of *equality*.

96. The foregoing definitions evidently apply to those instances only, in which the consequent of a ratio is contained a certain number of times in the antecedent, or in which the magnitude of the ratio may be expressed by some definite fraction. It does not therefore comprehend such ratios as $\sqrt{2} : 5$; $\sqrt{3} : \sqrt[3]{7}$; $4 : \sqrt[4]{10}$; &c. &c.; where the values of the quantities $\sqrt{2}$, $\sqrt{3}$, $\sqrt[3]{7}$ &c. can only be expressed in decimal fractions which do not terminate. The ratio which exists between quantities of this latter kind, when the radical

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(a) In expressing the ratio of two quantities, the word "*to*" is generally supplied by two dots; thus, the ratio of "*a* to *b*" is expressed by "*a* : *b*."

quantity is expressed by a decimal fraction, is called their *approximate* ratio.

97. *Proportion* consists in the *equality of ratios*; thus, since 4 is contained in 12, the same number of times that 6 is in 18, the ratio of $12 : 4$ is said to be equal to the ratio of $18 : 6$, or, in other words, that $12 : 4 :: 18 : 6$.[b] Of the four terms of which every proportion consists, the first and last terms are called the *extremes*, and the second and third the *means* of that proportion.

98. If there be a set of quantities related together in the following manner, viz. $a : b :: b : c :: c : d :: d : e$, &c. where the consequent of every preceding ratio is the antecedent of the following one, then the quantities a, b, c, d, e, &c. are said to be in *continued* proportion; and if only *three* quantities be concerned, as in the proportion $a : b :: b : c$, then b is said to be a *mean proportional* between the two extremes a and c.

99. Since the proportion $a : b :: c : d$ expresses the equality of the ratios $a : b$ and $c : d$; and since the magnitude of the ratio $a : b$ is measured by the fraction $\frac{a}{b}$, and that of the ratio $c : d$ by the fraction $\frac{c}{d}$, it follows that $\frac{a}{b} = \frac{c}{d}$, or, "that when "four quantities are proportional, the quotient of the first "divided by the second, is equal to the quotient of the third "divided by the fourth;" and *vice versa*, if "there be four "quantities a, b, c, d, such, that $\frac{a}{b} = \frac{c}{d}$, then those four quan- "tities are proportional, or $a : b :: c : d$."

XXVIII.

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[b] In stating a proportion, the words "*is to*" and "*to*" are generally supplied by two dots, and the words "*so is*" by four dots; thus, the proportion "a is to b so is c to d," is expressed by "$a : b :: c : d$."

XXVIII.

On the Comparison and Composition of Ratios.

100. *On the comparison of Ratios.*

I. Since the ratio of $a : b$ may be expressed by the fraction $\frac{a}{b}$, let the numerator and denominator of this fraction be multiplied by any quantity m (m being either *integral* or *fractional*), then $\frac{ma}{mb}=\frac{a}{b}$, and $\therefore$ the ratio of $ma : mb$ is the same with the ratio of $a : b$; from which we infer, that "if the terms "of a ratio be multiplied or divided by the same quantity, it "does not alter the value of the ratio." From hence also it appears, that a ratio is reduced to its *lowest terms* by dividing its antecedent and consequent by their greatest common measure.

II. "Ratios are compared together by reducing the fractions by which their values are respectively represented, to a "common denominator." Thus, the ratio of $8 : 5$ is represented by the fraction $\frac{8}{5}$, and the ratio of $9 : 6$ by the fraction $\frac{9}{6}$; reduce these fractions to others of the same value having a common denominator, and they become $\frac{48}{30}$ and $\frac{45}{30}$ respectively; and since $\frac{48}{30}$ is greater than $\frac{45}{30}$, the ratio $8 : 5$ is greater than the ratio of $9 : 6$.

III. "A ratio of greater inequality is *diminished*, and a ratio "of lesser inequality is *increased*, by *adding* the same quantity "to both its terms." Let $a+b : a$ represent a ratio of *greater inequality*, and let x be added to each of its terms, and it becomes the ratio of $a+b+x : a+x$. Now the ratio of $a+b : a=\frac{a+b}{a}$, and that of $a+b+x : a+x=\frac{a+b+x}{a+x}$; let these fractions be reduced to others of the same value having a common denominator, and they become $\frac{a^2+ab+ax+bx}{a(a+x)}$ and $\frac{a^2+ab+ax}{a(a+x)}$ respectively; and since $a^2+ab+ax+bx$ is evidently greater than $a^2+ab+ax$, the ratio of $a+b : a$ is greater than the ratio of $a+b+x : a+x$; i.e. the ratio of $a+b : a$ has been *diminished* by adding x to each of its terms. Next, let $a-b : a$ represent a ratio of *lesser inequality;* then proceeding with the fractions $\frac{a-b}{a}$ and $\frac{a-b+x}{a+x}$, as in the former instance, the resulting fractions are $\frac{a^2-ab+ax-bx}{a(a+x)}$ and $\frac{a^2-ab+ax}{a(a+)}$; and since $a^2-ab+ax-bx$ is less than $a^2-ab+ax$, the ratio of $a-b : a$ is less than the ratio of $a-b+x : a+x$, and consequently the ratio of $a-b : a$ has been *increased* by adding x to each of its terms. In the same manner it might be shewn that "a ratio of greater inequality is *increased,* and a ratio of lesser inequality is *diminished,* by "*subtracting* the same quantity from each of its terms."

101. *On the composition of Ratios.*

I. Ratios are compounded together by multiplying their antecedents together for a *new* antecedent, and their consequents together for a *new* consequent. Thus, if the ratio of $a : b$ be compounded with the ratio of $c : d$, the resulting ratio is that of $ac : bd$; or if the ratios 4 : 3; 5 : 2; and 7 : 1, be compounded together, there results the ratio of $4 \times 5 \times 7 : 3 \times 2 \times 1$, or of 140 : 6, or (dividing each term by 2) of 70 : 3.

II. If the *same* ratio be compounded with itself *once, twice, thrice,* &c. the resulting ratios are those of $a^2 : b^2$; $a^3 : b^3$; $a^4 : b^4$, &c. &c. The ratio of $a^2 : b^2$ is called the *duplicate* ratio of $a : b$; $a^3 : b^3$ the *triplicate*; $a^4 : b^4$ the *quadruplicate;* &c. &c.; and as these ratios receive their denominations from the *indices* of the several powers of a and b, the ratio of $\sqrt{a} : \sqrt{b}$ is called the *subduplicate* ratio of $a : b$; the ratio of $\sqrt[3]{a} : \sqrt[3]{b}$, the *subtriplicate*; &c. &c.

III. "If a set of ratios, whereof the consequent of the preceding ratio is the same with the antecedent of the succeeding "one, be compounded together, the resulting ratio is that of "the *first antecedent* to the *last consequent*." Thus, when the ratios of $a : b$; $b : c$; $c : d$; $d : e$; &c. are compounded together, the resulting ratio is that of $abcd$ &c.; $bcde$ &c. or (dividing by bcd) that of $a : e$ or of the *first antecedent* : the *last consequent*.

IV. "A ratio of *greater inequality* compounded with another "ratio, *increases* it; and a ratio of *lesser inequality* com"pounded with another ratio, *diminishes* it." Thus, let $1+n : 1$ represent a ratio of greater inequality, and let it be compounded with the ratio $a : b$, the resulting ratio is that of $a+na : b$, which is evidently *greater* than the ratio of $a : b$; on the other hand, let $1-n : 1$ represent a ratio of lesser inequality, and let it be compounded with the ratio of $a : b$, then the resulting ratio is that of $a-na : b$, which is evidently *less* than the ratio of $a : b$.

EXAMPLES.

Ex. 1. Reduce the ratio of 360 : 315, and 1595 : 667, to their lowest terms.

Ex. 2. Reduce the ratio of $a^3+2a^2x : a^3$ to its lowest terms.

Ex. 3. Which is the *greatest*, the ratio of 16 : 15, or that of 17 : 14?

Ex. 4. Which is the *least* of the three ratios, 20 : 17, 22 : 18, or 25 : 23? and which is the *greatest* of the three ratios 8 : 7; 6 : 5; and 10 : 9?

Ex. 5. Which is the *greatest*, the ratio of $a+2 : \frac{1}{2}a+4$, or that of $a+4 : \frac{1}{2}a+5$?

ANSWER, The ratio of $a+4 : \frac{1}{2}a+5$.

Ex. 6. Compound together the ratios of 11 : 3, 7 : 2, and 5 : 9. ANSWER, 385 : 54.

Ex. 7. Compound together the ratios of 15 : 12, 6 : 7, and 9 : 4; and then reduce the resulting ratio to its *lowest terms*.

ANSW. 135 : 56.

Ex. 8. Express in the *simplest* terms the ratio compounded of $a^2-x^2 : a^2$, $a+x : b$, and $b : a-x$.

ANSW. $(a+x)^2 : a^2$.

Ex. 9. If the ratios of $x+y : a$, $x-y : b$, and $b : \frac{x^2-y^2}{a}$, be compounded together, shew that the resulting ratio is a ratio of *equality*.

Ex. 10. If the ratios of $3a+2 : 6a+1$, and of $2a+3 : a+2$, be compounded together, is the resulting ratio a ratio of *greater* or *lesser* inequality?

ANSW. A ratio of *greater* inequality.

Ex. 11. What are the *least* numbers in the ratio compounded of the three following ratios, viz. the ratio of 7 : 5, the *duplicate* ratio of 4 : 9, and the *triplicate* ratio of 3 : 2?

ANSW. 14 and 15.

Ex. 12. Compound the *subduplicate* ratio of $x^2 : y^2$, with the *quadruplicate* ratio of $\sqrt{x} : \sqrt{y}$.

ANSW. $x^3 : y^3$.

XXIX.

On Proportion.

102. The most useful Theorems relating to proportional quantities are the following.

TH. 1. "If four quantities be proportional, the product of "the extremes will be equal to the product of the means;" for let $a : b :: c : d$, then, by Art. 99, $\frac{a}{b}=\frac{c}{d}$, $\therefore\ ad=bc$. From hence also it follows, "that if any three terms of a proportion be known, the fourth may be found;" for, from the equation $ad=bc$, we have $a=\frac{bc}{d}$; $b=\frac{ad}{c}$; $c=\frac{ad}{b}$; and $d=\frac{bc}{a}$.

Th. 2. The converse of the foregoing Theorem is also true; viz. "If the product of any two quantities be equal to the "product of two others, those four quantities will constitute "a proportion, provided that the terms of one product be made "the *means*, and the terms of the other product be made the "*extremes*, of such proportion." Thus, if the four quantities a', b', c', d' be such that $a'd'=b'c'$; then (dividing by $b'd'$) $\frac{a'}{b'}=\frac{c'}{d'}$; $\therefore$ by Art. 99, $a' : b' :: c' : d'$.

Th. 3. "If three quantities be proportional, the product of "the two extremes is equal to the square of the mean;" for, if $a : b : : b : c$, then, by Theor. 2, $ac=b^2$. From hence also it follows, that "a mean proportional between any two quantities is equal to the square root of their product;" for let x be a mean proportional between a and c, then $a : x :: x : c$, $\therefore x^2=ac$, and $x=\sqrt{ac}$.

Th. 4. "If four quantities be proportional, they will also be proportional, when taken *inversely* or *alternately*;" thus, if $a : b :: c : d$, then $\frac{a}{b}=\frac{c}{d}$; *invert* the fractions, then $\frac{b}{a}=\frac{d}{c}$; $\therefore b : a : d : c$. Again, since $ad=bc$, then (dividing by cd) we have $\frac{ad}{cd}=\frac{bc}{cd}$, or $\frac{a}{c}=\frac{b}{d}$; $\therefore a : c : b : d$, and $b : d :: a : c$.

Th. 5. "If there be six proportional quantities, and the "first be to the second as the third to the fourth; and the third "to the fourth as the fifth to the sixth; then will the first "be to the second as the fifth to the sixth." For let $a : b :: c : d$, and $c : d :: e : f$; then $\frac{a}{b}=\frac{c}{d}$; and $\frac{c}{d}=\frac{e}{f}$; $\therefore \frac{a}{b}=\frac{e}{f}$, or by Art. 99, $a : b :: e : f$.

TH. 6. "If four quantities be proportional, then the *sum* "or *difference* of the first and second will be to the second "as the *sum* or *difference* of the third and fourth is to the "fourth." For let $a : b :: c : d$, then $\frac{a}{b}=\frac{c}{d}$; *add* or *subtract* 1 from each side of the equation; then $\frac{a}{b}\pm 1=\frac{c}{d}\pm 1$, $\therefore \frac{a\pm b}{b}=\frac{c\pm d}{d}$, consequently, by Art. 99, $a\pm b : b :: c\pm d : d$.

TH. 7. "If four quantities be proportional, the *first* is to "the *sum* or *difference* of the first and second, as the *third* to "the *sum* or *difference* of the third and fourth." For by THEOR. 6. $a\pm b : b :: c\pm d : d$, and alternately $a\pm b : c\pm d :: b : d$; but by THEOR. 4, $b : d :: a : c$; hence, by THEOR. 5, $a\pm b : c\pm d :: a : c$, and alternately $a\pm b : a :: c\pm d : c$, $\therefore$ *inversely* $a : a\pm b :: c : c\pm d$.

TH. 8. "If four quantities be proportional, then the *sum* "of the first and second is to their *difference*, as the *sum* of "the third and fourth is to *their* difference." For by THEOR. 6, $\frac{a+b}{b}=\frac{c+d}{d}$, and $\frac{a-b}{b}=\frac{c-d}{d}$; *invert* the two last fractions, then $\frac{b}{a-b}=\frac{d}{c-d}$; hence $\frac{a+b}{b}\times\frac{b}{a-b}=\frac{c+d}{d}\times\frac{d}{c-d}$, or $\frac{a+b}{a-b}=\frac{c+d}{c-d}$; $\therefore$ by Art. 99, $a+b : a-b :: c+d : c-d$.

TH. 9. "If four quantities be proportional, and any *equi-* "*multiple* or *equal parts* whatever be taken of the first and "second, and also of the third and fourth; then will the "resulting quantities, taken in the same order, be still propor- "tional." For let $a : b :: c : d$; then, by CASE I, Art. 100, the ratio of $ma : mb$ is the same with the ratio of $a : b$; and for the same reason, the ratio of $nc : nd$ is the same with the ratio of $c : d$; hence (Art. 97) $ma : mb :: nc : nd$, where m and n may be any quantities whatever, either *integral* or *fractional.*

Th. 10. The same theorem is true "if any *equimultiple* or "*equal parts* whatever be taken of the *first and third,* and also "of the *second and fourth;*" for since $\frac{a}{b}=\frac{c}{d}$, multiply each side of the equation by $\frac{m}{n}$, then $\frac{ma}{nb}=\frac{mc}{nd}$, $\therefore ma : nb :: mc : nd$, where m and n may be any quantities whatever, either *integral* or *fractional.*

Th. 11. "If four quantities be proportional, any *powers* or *roots* of those quantities will also be proportional." For since $\frac{a}{b}=\frac{c}{d}$, we have $\frac{a^n}{b^n}=\frac{c^n}{d^n}$, $\therefore a^n : b^n : c^n : d^n$, where n may be any number either *integral* or *fractional.*

Th. 12. "If the corresponding terms of two sets of pro-"portionals be multiplied together, or divided by each other, the "resulting quantities taken in order will still be proportional." Thus let

$$\left.\begin{matrix} a:b::c:d \\ \text{and} \\ e:f::g:h \end{matrix}\right\} \begin{matrix} \text{then } \frac{a}{b}=\frac{c}{d} \\ \\ \text{and } \frac{e}{f}=\frac{g}{h} \end{matrix} \left.\vphantom{\begin{matrix} a \\ b \\ c \end{matrix}}\right\} \text{hence } \frac{ae}{bf}=\frac{cg}{dh} \text{ or } ae : bf :: cg : dh.$$

Again, by Th. 1, $ad=bc$, and $eh=fg$; $\therefore \frac{ad}{eh}=\frac{bc}{fg}$; hence, by Th. 2, $\frac{a}{e} : \frac{b}{f} : \frac{c}{g} : \frac{d}{h}$.

Th. 13. "If there be two rows of proportional quantities, "whereof the *second* and *fourth* of the first row are the same "with the *first* and *third* of the second row, then will the "remaining quantities, taken in order, be proportional;" thus, let $a : b :: c : d$

and $b : e :: d : f$, then by THEOR. 12, $ab : be :: cd : df$, or (reducing each ratio to its lowest terms) $a : e :: c : f$; This is the *ex æquali* proportion of Euclid.

TH. 14. "If there be a set of proportional quantities, "$a : b :: c : d :: e : f :: g : h$ &c. &c., then will the *first* be "to the *second* as the *sum of all the antecedents* to the *sum of* "*all the consequents.*"

$$\begin{array}{rll}
\text{For } a : b & :: c & : d, \\
\text{and } \textit{alternately}, \ a : c & :: b & : d. \\
\text{Hence by Theor. 7, } a : a+c & :: b & : b+d; \\
\therefore \textit{alternately}, \ a : b & :: a+c & : b+d.
\end{array}$$

$$\begin{array}{rll}
\text{Again, } a : b & :: e & : f; \\
\therefore \text{by Theor. 5. } a+c : b+d & :: e & : f, \\
\textit{alternately}, \ a+c : e & :: b+d & : f, \\
\text{By Theor. 6. } a+c+e : e & :: b+d+f & : f; \\
\therefore \textit{alternately}, \ a+c+e : b+d+f & :: e & : f. \\
\text{But } a : b & :: e & : f; \\
\therefore a : b & :: a+c+e & : b+d+f;
\end{array}$$

And so on for any number of these proportions.

TH. 15. "If there be a set of quantities, $a, b, c, d, e,$ &c. in *continued* proportion;

"then $a : c :: a^2 : b^2$, or in the *duplicate* ratio of $a : b$;
"$a : d :: a^3 : b^3$, or in the *triplicate* ratio of $a : b$;
"$a : e :: a^4 : b^4$, or in the *quadruplicate* ratio of $a : b$;"
&c. &c. &c. &c.

For, Art. 98. $a : b :: b : c :: c : d :: d : e ::$ &c. &c.

$$\therefore a : b :: b : c;$$

and by THEOR. 11, $a^2 : b^2 :: b^2 : c^2$.

But by THEOR. 3, $b^2 = ac$;

$$\therefore a^2 : b^2 :: ac : c^2,$$

$$:: a : c, \text{ or } a : c :: a^2 : b^2.$$

Again, $a : c :: a^2 : b^2$,

but $c : d :: a : b$;

$\therefore$ by THEOR. 12, $a : d :: a^3 : b^3$.

Moreover, $a : d :: a^3 : b^3$,

but $d : e :: a : b$;

$\therefore$ by THEOR. 12, $a : e :: a^4 : b^4$.

&c. &c. &c. &c.

103. The following Examples are intended to illustrate the use of the foregoing Theorems.

EXAMPLE 1.

To divide the number 60 into two such parts, that the *product* shall be the *sum of the squares* $:: 2 : 5$.

Let x = one part;

then $60-x$ = the other part,

$(60-x) \times x = 60x - x^2$ = the *product*,

and $x^2 + (60-x^2) = 2x^2 + 3600 - 120x$ = *sum of the squares.*

Hence, by the question, $60x - x^2 : 2x^2 + 3600 - 120x :: 2 : 5$;

$\therefore$ by THEOR. 1, $(60x - x^2) \times 5 = (2x^2 + 3600 - 120x) \times 2$,

or $300x - 5x^2 = 4x^2 + 7200 - 240x$;

by *transposition* and *division*, $x^2 - 60x = -800$;

$\therefore x^2 - 60x + 900 = 900 - 800 = 100$,

and $x - 30 = \pm 10$;

or $x = 30 \pm 10 = 40$ *or* 20 the [parts required.

Ex. 2.

The number 20 is divided into two parts, which are to each other in the *duplicate* ratio of 3 : 1. Find a *mean proportional* between those parts.

Let $x=$ *greater* part,
then $20-x=$ *lesser* part;
∴ by the question, $x : 20-x :: 3^2 : 1^2 :: 9 : 1$.
Hence, by Theor. 1, $x=180-9x$,
or $10x=180$;
∴ $x=18$ *greater* part,
and $20-x=20-18=2$ *lesser* part.

By Theor. 3, a *mean proportional* between 18 and 2 is equal to $\sqrt{18\times 2}=\sqrt{36}=6$ the number required.

Ex. 3.

If $(a+x)^2 : (a-x)^2 :: x+y : x-y$, shew that $a : x :: \sqrt{2a-y} : \sqrt{y}$.
By *expansion*, $a^2+2ax+x^2 : a^2-2ax+x^2 :: x+y : x-y$.
By Theor. 8, $2a^2+2x^2 : 4ax :: 2x : 2y$.
Divide by 2, then $a^2+x^2 : 2ax :: x : y$;
∴ by Theor. 1, $(a^2+x^2)\times y=2ax\times x=2a\times x^2$.
Hence, by Theor. 2, $a^2+x^2 : x^2 :: 2a : y$.
By Theor. 6, $a^2 : x^2 :: 2a-y : y$;
and by Theor. 11, (n being $\frac{1}{2}$) $a : x :: \sqrt{2a-y} : \sqrt{y}$.

Ex. 4.

If $x : y$ in the *triplicate* ratio of $a : b$, and $a : b :: \sqrt[3]{c+x} : \sqrt[3]{a+y}$, shew that $dx=cy$.
Since $x : y :: a^3 : b^3$,
and by Theor. 11, $a^3 : b^3 :: c+x : d+y$;
∴ by Theor. 5, $x : y :: c+x : d+y$,
or $c+x : d+y :: x : y$,
and by Theor. 4. $c+x : x :: d+y : y$;
∴ Theor. 6, $c : x :: d : y$;
and by Theor. 1, $dx=cy$.

Ex.

Ex. 5.

There are two numbers whose *product* is 24, and the *difference of their cubes* : *cube of their difference* :: 19 : 1. What are the numbers ?

Let x = *greater* number,
and y = *lesser* number.

Then, by the question, $xy=24$,
and $x^3-y^3 : (x-y)^3 :: 19 : 1$.

By *expansion*, $x^3-y^3 : x^3-3x^2y+3xy^2-y^3(\overline{x-y}|^3) :: 19 : 1$.
By Theor. 8, $3x^2y-3xy^2 : (x-y)^3 :: 18 : 1$,
or $3xy\times(x-y) : (x-y)^3 :: 18 : 1$.
Divide by $x-y$, then $3xy : (x-y)^2 :: 18 : 1$;
but $xy=24$; $\therefore 72 : (x-y)^2 :: 18 : 1$.

Hence, by Theor. 1, $18\times(x-y)^2=72$,
or $(x-y)^2=4$;
$\therefore x-y=2$.

Again, $x^2-2xy+y^2=4$,
and $4xy\ \ =96$.

$$\therefore x^2+2xy+y^2=100,$$

$$\left.\begin{array}{l}\text{or } x+y=10,\\ \text{but } x-y=2;\end{array}\right\}\begin{array}{l}\therefore x=\frac{12}{2}=6,\\ \quad\ y=\frac{8}{2}=4.\end{array}$$

Ex. 6. To divide the number 24 into two such parts, that their *product* shall be to the *sum of their squares* :: 3 : 10.

Answer, 18 and 6.

Ex. 7. There are two numbers which are to each other as 3 : 2. If 6 be *added to the greater*, and *subtracted from the lesser*, the *sum* and *remainder* will be to each other :: 3 : 1. What are the numbers ?

Answ. 24 and 16.

Ex. 8. There are two numbers which are to each other in the *duplicate* ratio of 4 : 3, and 24 is a *mean proportional* between them. What are the numbers?

ANSWER, 32 and 18.

Ex. 9. If $\frac{a^2-x^2}{b}=4a$; shew that $a+x : 2a :: 2b : a-x$.

Ex. 10. If $x^2 : y^2 :: 36 : 25$, and $2x+y : x+2$ in a ratio compounded of the ratios of 17 :: 2 and 2 : 7; what are the numbers? ANSW. 12 and 10.

Ex. 11. If $a+x : a-x :: 9 : 5$, shew (by Theor. 8.) that $a : x :: 7 : 2$.

Ex. 12. There are two numbers whose product is 135, and the *difference of their squares* is to the *square of their difference* :: 4 : 1. What are the numbers?

ANSW. 15 and 9.

XXX.

On Variable Quantities.

104. If the quantities under consideration be of a *variable* nature, then their relation to each other must be estimated in the following manner.

1. Let A and B be two variable quantities so related to each other, that whilst the value of A is changed to a, the value of B is changed to b; then if these two quantities A and B always bear the same ratio to each other, i.e. if $A : B :: a : b$ (or by Theor. 4. of Prop^n. $A : a :: B : b$) throughout the whole period of their variation, they are said to vary *directly* as each other.

EXAM. Suppose a body to move uniformly along at the rate of 3 feet in one second of time; then in the *first* second it would describe 3 feet, in *two* seconds 6 feet, in *three* seconds 9 feet, &c. &c.; hence, whilst the time varies through

1, 2,

1, 2, 3, 4, &c. seconds, the space varies through 3, 6, 9, 12, &c. feet; but the numbers 3, 6, 9, &c. are respectively in the same ratio with the numbers 1, 2, 3, &c. When a body moves uniformly, therefore, "the space varies *directly* as the time."

II. If the relation between A and B be such, that whilst A by *increasing* is changed to a, and B by *decreasing* is changed to b, in such manner, that $A : a :: \frac{1}{B} :: \frac{1}{b}$ (or) $:: b : B$ throughout the whole period of their variation, then A is said to vary *inversely* as B.

Exam. The area of a triangle is equal to half the rectangle contained by its base and perpendicular altitude; if, therefore, the *form* of the triangle be changed whilst its *area* remains the same, it is evident that as its *altitude* increases its *base* must decrease. Let A and B represent its altitude and base at any one period of its variation, and a and b its altitude and base at any other period, then $\frac{A\times B}{2}=\frac{a\times b}{2}$ or $A\times B=a\times b$, $\therefore$ (by Th. 2. Prop^n.) $A : a :: b : B :: \frac{1}{B} : \frac{1}{b}$, i.e. "the "altitude of a triangle whose area is given varies *inversely* "as its base, and vice versâ."

III. If there be three variable quantities A, B, C, whose relation to each other is such, that whilst B is changed to b, and C to c, A is changed in the *compound* ratio of the change of B and C; i.e. if $A : a$ in the ratio compounded of the ratios of $B : b$ and $C : c$, or (Art. 101, I.) $A : a :: BC : bc$, then A is said to vary as B and C *conjointly*.

Exam. Let A represent the *area*, B the *base*, and C the *perpendicular altitude* of a triangle; and when the form of this triangle is changed, let a represent the *area*, b the *base*, and c the *altitude* at any period of its variation; then $A=\frac{BC}{2}$

and $a=\frac{bc}{2}$, $\therefore A : a :: \frac{BC}{2} : \frac{bc}{2} :: BC : bc$, or "the *area* "of a triangle varies as its *base and perpendicular altitude* "*conjointly*."

IV. If the relation between the three quantities A, B, C be such, that when A is changed to a, B to b, and C to c, $B : b$ in the ratio compounded of the ratios of $A : a$ and $\frac{1}{C} : \frac{1}{c}$ or (Art. 101, I.) $B : b :: \frac{A}{C} : \frac{a}{c}$, then B is said to vary *directly* as A, and *inversely* as C.

Exam. Let A, B, C, a, b, c represent the same quantities as in the last Example, then since $A=BC$, $B=\frac{A}{C}$; and since $a=bc$, $b=\frac{a}{c}$. Hence $B : b :: \frac{A}{C} : \frac{a}{c}$, i.e. "when the area "of a triangle is changed, the base will vary as the area "*directly*, and as the perpendicular altitude *inversely*."

105. These several relations of variable quantities are often more briefly expressed by placing the mark $\propto$ between them; thus

$A : a :: B : b$, or "A varies as B," is expressed by . . $A \propto B$.

$A : a :: \frac{1}{B} : \frac{1}{b}$, or "$A$ varies *inversely* as B," by . . . $A \propto \frac{1}{B}$.

$A : a :: BC : bc$, or "A varies as B and C *conjointly*," by $A \propto BC$.

$B : b :: \frac{A}{C} : \frac{a}{c}$ {or "B varies *directly* as A, and *inversely* as C," . . . . } by . . $B \propto \frac{A}{C}$.

This notation is made use of in the following Theorems.

Th. 1. "If one quantity varies as another, it will also vary as any *multiple part*, *power* or *root* of the other." Thus, let $A \propto B$, then $A : a :: B : b$; multiply the last ratio by m, then (Art. 100, I.) $A : a :: mB : mb$, $\therefore$ (Art. 104, I.)

$A \propto mB$, where m may be any number either *integral* or *fractional*. Again, since $A : a :: B : b$, (by Th. 11, Prop^n.) $A^n : a^n :: B^n : b^n$; $\therefore A^n \propto B^n$, where n may be any number *integral* or *fractional*.

Th. 2. "If one quantity varies as another, and each of "them be multiplied or divided by any quantity variable or "invariable, then will the products or quotients, thus arising, "vary as each other." Thus, let $A \propto B$, then $A : a :: B : b$; let m be an *invariable* quantity, and multiply all the terms of the proportion by it, then $mA : ma :: mB : mb$, $\therefore mA \propto mB$. Let C be a *variable* quantity, then we have

$$\left.\begin{array}{l} A : a :: B : b \\ \quad\text{and} \\ C : c :: C : c \end{array}\right\} \therefore \text{by Th. 12, Prop}^n. \left\{\begin{array}{l} AC : ac :: BC : bc, \text{ or } AC \propto BC; \\ \quad\text{and} \\ \frac{A}{C} : \frac{a}{c} :: \frac{B}{C} : \frac{b}{c}, \text{ or } \frac{A}{C} \propto \frac{B}{C}. \end{array}\right.$$

Cor. 1. From hence it follows, that "if one quantity "varies as two others jointly, then either of those quantities "varies as the first *directly* and the other *inversely*." Thus let $A \propto BC$, then dividing each by C, $B \propto \frac{A}{C}$ or "as A directly and C inversely;" divide by B, then $C \propto \frac{A}{B}$ or "as A directly and B inversely."

Cor. 2. "If the product of two quantities be invariable, then those quantities vary *inversely* as each other." For let $A \times B = m$, then $A = \frac{m}{B}$ which varies as $\frac{1}{B}$, and $B = \frac{m}{A}$ which varies as $\frac{1}{A}$, m being a constant quantity.

Th. 3. "If one quantity varies as a second, and the second as a third, then will the first quantity vary as the third." For let $A \propto B$, then $A : a :: B : b$; and let $B \propto C$, then $B : b :: C : c$; $\therefore$ by Th. 5, Prop^n, $A : a :: C : c$; hence $A \propto C$.

Th. 4. "If any two quantities vary as a third, then will "their *sum* or *difference* or the *square root of their product* "vary as the third." Thus let $A\propto C$ and $B\propto C$, then, by Th. 3, $A\propto B$; $\therefore A : a :: B : b$ or $A : B :: a : b$, and by Th. 6. Prop^{n}. $A\pm B : B :: a\pm b : b$ or $A\pm B : a\pm b :: B : b$; but since $B\propto C$, $B : b :: C : c$, hence $A\pm B : a\pm b :: C : c$, or $A\pm B\propto C$.

Again, since

$$\left.\begin{array}{l} A : a :: C : c \\ \text{and } B : b :: C : c \end{array}\right\} \begin{array}{l} \text{by Th. 12, Prop}^{n}.\ AB : ab :: C^2 : c^2, \\ \text{and Th. 11, Prop}^{n}.\ \sqrt{AB} : \sqrt{ab} :: C : c. \end{array}$$

Hence $\sqrt{AB}\propto C$.

Th. 5. "If the *square of the sum* of two quantities varies "as the *square of their difference*, then the *sum of their* "*squares* varies as their *product*." For let $(A+B)^2\propto(A-B)^2$, then

$$(A+B)^2 : (a+b)^2 :: (A-B)^2 : (a-b)^2,$$

$$\text{or } (A+B)^2 : (A-B)^2 :: (a+b)^2 : (a-b)^2.$$

$$\left.\begin{array}{l} \text{By Expansion, and} \\ \text{Th. 8, Prop}^{n}. \end{array}\right\} \begin{array}{l} 2A^2+2B^2 : 4AB :: 2a^2+2b^2 : 4ab, \\ \text{or } A^2+B^2 : 2AB :: a^2+b^2 : 2ab; \end{array}$$

$$\therefore A^2+B^2 : a^2+b^2 :: 2AB : 2ab :: AB : ab.$$

Hence $A^2+B^2\propto AB$.

Th. 6. "If there be two sets of quantities, A, B, C, D, "&c. and P, Q, R, S, &c. which vary as each other re- "spectively, viz. $A\propto P$, $B\propto Q$, then will the *products* of "those quantities vary as each other." For, let a, b, c, &c. p, q, r, &c. be corresponding values of A, B, C, &c. P, Q, R, &c. then,

$$\begin{array}{l} \text{since } A\propto P,\ A : a :: P : p \\ \quad \ldots\ B\propto Q,\ B : b :: Q : q \\ \quad \ldots\ C\propto R,\ C : c :: R : r \\ \qquad \text{\&c.} \qquad \text{\&c.} \end{array}$$

$\therefore$ By Cor. to Theor. 12, Prop^{n}.

$$ABC\ \text{\&c.} : abc\ \text{\&c.} :: PQR\ \text{\&c.} : pqr\ \text{\&c.}$$

Hence ABC &c. $\propto PQR$ &c.

TH. 7. "If any quantity A depends upon a set of quantities, P, Q, R, S, in such a manner, that if Q, R, S are "constant, $A \propto P$; if P, R, S are constant, $A \propto Q$; &c. &c. "then if they *all* vary, A will vary as their *product*."

For let A be changed

to x, by the variation of P to p, the rest being *constant*,
from x to y . . . . . of Q to q, ...
from y to z . . . . . of R to r, ...
from z to a . . . . . of S to s . . . .

Then, when *all* vary, we have

$$\left.\begin{array}{l} A : x :: P : p \\ x : y :: Q : q \\ y : z :: R : r \\ z : a :: S : s \end{array}\right\}$$

Hence, by composition of ratios, $A : a :: PQRS : pqrs$ or $A \propto PQRS$; and the Theorem would evidently be true, whatever be the number of quantities P, Q, R, S, &c.

TH. 8. "If one quantity varies as another, it is *equal* "to that quantity multiplied into some *constant* quantity; "and the value of this constant quantity will be known, if the "actual relation between the two variable quantities at some "given period of their increase or decrease be known." For let $A \propto B$, then $A : a :: B : b$ or $A : B :: a : b$, i.e. the ratio of $A : B$ is always the *same* through the whole period of their variation; let this ratio be that of $m : 1$, then $A : B :: m : 1$, and $A = mB$, or $m = \frac{A}{B}$. If therefore the corresponding values of A and B at any period of their variation be known, the value of m will be known.

EXAM. The *space* described by a body descending perpendicularly near the surface of the Earth varies as the square of the *time*; let the *space* $= S$, the corresponding *time* $= T$, then by this Theorem $S = mT^2$; now it is known by experiment, that a body falls through a space of about 16 feet in the *first* second of its fall; hence, when $S=16$, $T=1$, $\therefore m=16$, and the general relation between the space and time of a body thus falling is $S=16T^2$.

Cor. Since $\frac{A}{B}=m$, it follows, "that if one quantity "varies as another, the fraction arising from dividing the "one quantity by the other, is a *constant* quantity."

CHAP. VII.

ON ARITHMETICAL AND GEOMETRICAL PROGRESSION.

XXXI.

DEFINITIONS.

106. If a series of quantities increase or decrease by the continual *addition* or *subtraction* of the same quantity, then those quantities are said to be in *Arithmetical* Progression. Thus the numbers 1, 2, 3, 4, 5, 6, &c. (which *increase* by the *addition* of 1 to each successive term), and the numbers 21, 19, 17, 15, 13, 11, &c. (which *decrease* by the *subtraction* of 2 from each successive term), are in arithmetical progression.

107 In general, if a represents the *first* term of any arithmetical progression, and b the *common difference*, then may the series itself be expressed by $a, a+b, a+2b, a+3b, a+4b,$ &c. which will evidently be an *increasing* or a *decreasing* one, according as b is *positive* or *negative*. In the foregoing series, the *coefficient* of b in the *second* term is *one*; in the *third* term is *two;* in the *fourth* is *three*, &c. i.e. the coefficient of b in any term is always *less by unity* than the number which denotes *the place of that term in the series*. Hence, if the number of terms in the series be denoted by (n), the nth or *last* term in the progression will be $a+(n-1)b$.

108. If a series of quantities increase or decrease by the continual *multiplication* or *division* of the same quantity, then those quantities are said to be in *Geometrical* Progression. Thus the numbers 1, 2, 4, 8, 16, &c. (which *increase* by the continual *multipli-*

multiplication of 2), and the numbers $1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}$, &c. (which *decrease* by the continued *division* of 3, or *multiplication* of $\frac{1}{3}$), are in Geometrical Progression.

109. In general, if a represents the *first term* of such a series, and r the *common multiple* or *ratio,* then may the series itself be represented by a, ar, ar^2, ar^3, ar^4, &c. which will evidently be an *increasing* or *decreasing* series, according as r is a *whole number* or a *proper fraction.* In the foregoing series, the *index of* r in any term is *less by unity* than the number which denotes *the place of that term in the series.* Hence, if the number of terms in the series be denoted by (n), the *last term* will be ar^{n-1}.

XXXII.

On Arithmetical Progression.

110. Let S be the *sum* of the series a, $a+b$, $a+2b$, $a+3b$, &c.; then

$$\begin{array}{l} \left(a\ldots\ldots\right)+\left(a+b\right)+\left(a+2b,\right.\ \&c.\ \left(\overline{a+(n-2)b}\right)+\left(\overline{a+(n-1)b}\right)=S, \\ \quad\text{and,} \\ \left(\overline{a+(n-1)b}\right)+\left(\overline{a+(n-2)b}\right)+\left(\overline{a+(n-3)b}\right)\ \&c.\ \left(a+b\right)+\left(a\ldots\ldots\right)=S, \end{array}$$

where the *lower* series is the same as the *upper* one, except that the order of the terms is inverted.

Add the two series together, by forming the terms which are written within the parentheses into one sum, and we have,

$$\overline{2a+(n-1)b}+\overline{2a+(n-1)b}+\overline{2a+(n-1)b}+\ \&c.\ \text{to } n \text{ terms}=2S,$$

or $\overline{2a+(n-1)b}\times n=2S;\ \therefore S=\overline{2a+(n-1)b}\times\frac{n}{2}$.(a)

(a) Since the sum of any two terms within the parentheses $=\overline{a+a+(n-1)b}=$ sum of *first* and *last* term, and since $S=\overline{2a+(n-1)b}\times\frac{n}{2}$, it appears that the sum of the series is equal to *the sum of the first and last terms (or of any two terms equally distant from the first and last terms), multiplied into half the number of terms.*

111. From the equation $\overline{2a+(n-1)b}.n=2S$, it appears that if any three of the four quantities a, b, n, S are given, the fourth may be found. For we have,

I. By Art. 110 . . . . $S=\overline{2a+(n-1)b}\times\frac{n}{2}$.

II. By actual multiplication, $2an+bn^2-bn=2S$;

$$\therefore 2an=2S-bn^2+bn,$$

$$\text{and } a=\frac{2S-bn^2+bn}{2n}.$$

III. . . . . Again, $bn^2-bn=2S-2an$

$$\text{or } (n^2-n)b=2S-2an;$$

$$\therefore b=\frac{2S-2an}{n^2-n}.$$

IV. To find n, we have, by transposition, $bn^2+2an-bn=2S$,

$$\text{or } bn^2+(2a-b)n=2S;$$

$$\therefore n^2+\frac{2a-b}{b}.n=\frac{2S}{b}.$$

Solve this *quadratic* equation, and it gives the value of n.

EXAMPLE 1.

Find the sum of the series 1, 3, 5, 7, 9, 11, &c. continued to 120 terms.

$$\text{Here }\left.\begin{aligned}a&=1,\\ b&=2,\\ n&=120;\end{aligned}\right\}\therefore S=\overline{2a+(n-1)b}\times\frac{n}{2}.$$

$$=\overline{2\times 1+(120-1)2}\times\frac{120}{2}.$$

$$=\overline{2+119\times 2}\times 60=240\times 60=14400.$$

Ex. 2.

Find the sum of the series 15, 11, 7, 3, −1, −5, &c. to 20 terms.

$$\text{Here }\left.\begin{aligned}a&=\ 15,\\ b&=-4,\\ n&=\ 20;\end{aligned}\right\}\therefore S=\overline{2a+(n-1)b}\times\frac{n}{2}.$$

$$=\overline{2\times 15+(20-1)\times\ -4}\times\frac{20}{2}.$$

$$=\ (30-19\times 4)\times 10.$$

$$=(30-76)\times 10=-46\times 10=-460.$$

Ex. 3.

Ex. 3.

Find the sum of 150 terms of the series $\frac{1}{3}, \frac{2}{3}, 1, \frac{4}{3}, \frac{5}{3}, 2, \frac{7}{3}$, &c.

$$\left.\begin{array}{r} \text{Here } a=\frac{1}{3}, \\ b=\frac{1}{3}, \\ n=150; \end{array}\right\} \begin{array}{l} \therefore S=\overline{2a+(n-1)b}\times\frac{n}{2}. \\ =\overline{2\times\frac{1}{3}+(150-1)\times\frac{1}{3}}\times\frac{150}{2}. \\ =\overline{\frac{2}{3}+\frac{149}{3}}\times 75=\frac{151}{3}\times 75=3775. \end{array}$$

Ex. 4.

The *sum* of an arithmetic series is 1240, *common difference* -4, and *number of terms* 20. What is the *first term?*

$$\left.\begin{array}{r} \text{Here } S=1240, \\ b=-4, \\ n=20; \end{array}\right\} \begin{array}{l} \therefore a=\frac{2S-bn^2+bn}{2n}. \\ =\frac{2480+1600-80}{40}=\frac{4000}{40}=100. \end{array}$$

Hence the series is 100, 96, 92, 88, &c.

Ex. 5.

The *sum* of an arithmetic series is 1455, the *first term* 5, and the *number of terms* 30. What is the *common difference?*

$$\left.\begin{array}{r} \text{Here } S=1455, \\ a=5, \\ n=30; \end{array}\right\} \begin{array}{l} \therefore b=\frac{2S-2an}{n^2-n}. \\ =\frac{2910-300}{900-30}=\frac{2610}{870}=3. \end{array}$$

Hence the series is 5, 8, 11, 14, &c.

Ex. 6.

The *sum* of an arithmetic series is 567, the *first term* 7, the *common difference* 2. What are the *number of terms?*

$$\left.\begin{array}{r} \text{Here } S=567, \\ a=7, \\ b=2; \end{array}\right\} \begin{array}{l} \therefore n^2+\frac{2a-b}{b}.n=\frac{2S}{b} \\ \text{is } n^2+\ \ 6n\ \ =\frac{1134}{2}\ \ =567, \end{array}$$

$$\begin{array}{l} \text{and } n^2+\ 6n+9 = 567+9=576; \\ \therefore n+\ 3 \qquad = \sqrt{576}\ =24, \\ \qquad\qquad \text{or } n=24-3\ =21. \end{array}$$

Ex. 7.

How much ground does a person pass over in gathering up 200 stones placed in a straight line, at intervals of 2 feet from each other; supposing that he brings each stone *singly* to a basket standing at the distance of 20 yards from the first stone, and that he starts from the spot where the basket stands?

It is evident that the space passed over by this person will be *twice* the sum of an arithmetic series, whose *first term* is 20 *yards* (i. e. 60 *feet*), *common difference* 2 feet, and *number of terms* 200.

$$\left.\begin{aligned} \text{Here } a&=60,\\ b&=2,\\ n&=200;\end{aligned}\right\} \quad \begin{aligned} \therefore S&=\overline{2a+(n-1)b}\times\frac{n}{2}.\\ &=\overline{120+398}\times 100.\\ &=518\times 100=51800 \text{ feet.}\end{aligned}$$

$$\text{Hence the distance required}=\overset{\text{feet.}}{103600}=\overset{\text{miles,}}{19}\ .\ \overset{\text{furlongs,}}{4}\ .\ \overset{\text{feet.}}{640}.$$

Ex. 8.

A traveller bound to a place at the distance of 198 miles, goes 30 miles the *first* day, 28 the *second*, 26 the *third*, and so on. In how many days will he arrive at his journey's end?

$$\left.\begin{aligned} \text{Here is given } a&=30,\\ b&=-2,\\ S&=198,\end{aligned}\right\} \text{to find the } \textit{number of terms}.$$

$$\text{Now } n^2+\frac{2a-b}{b}.n=\frac{2S}{b}\ ;\ \therefore n^2-31n=-\frac{2\times 198}{2}=-198,$$

$$\text{and } n^2-31n+\frac{961}{4}=-198+\frac{961}{4}=\frac{169}{4}.$$

$$\text{Hence } n-\frac{31}{2}=\pm\frac{13}{2}, \text{ and } n=\frac{31-13}{2}=9.$$

The traveller, therefore, arrives at his journey's end in 9 days.

Ex. 9.

There are a certain number of quantities in arithmetic progression, whose *common difference* is 2, and whose *sum* is equal to eight times their *number*; moreover, if 13 be added to the *second* term, and this sum be divided by the *number of terms*, the quotient will be equal to the *first term*. What are the numbers?

$$\left.\begin{array}{l}\text{Let the } \textit{first term}=x,\\ \text{and the } \textit{number of terms}=y;\end{array}\right\}\begin{array}{l}\text{then the } \textit{second} \text{ term will be } x+2,\\ \ldots \text{ the } \textit{last} \text{ term } \ldots\ldots\ x+(y-1)\times 2.\end{array}$$

In the expression $\overline{2a+(n-1)b}\times\frac{n}{2}$, substitute x for a, 2 for b, and y for n, and it becomes $\overline{2x+(y-1)2}\times\frac{y}{2}$ $(=xy+y^2-y)$, for the *sum* of the series.

$$\text{By the question, } xy+y^2-y=8y, \text{ or } y=9-x,$$

$$\text{and } \frac{x+2+13}{y}=x.$$

$$\text{Hence, } \frac{x+2+13}{9-x}=x, \text{ or } x^2-8x=-15;$$

$$\therefore x^2-8x+16=16-15=1,$$

$$\text{and } x-4=\pm 1; \therefore x=5 \text{ or } 3,$$

$$y=9-x \quad =4 \text{ or } 6.$$

From which it appears that there are *two* sets of numbers which will answer the conditions required; viz. 5, 7, 9, 11, or 3, 5, 7, 9, 11, 13.

Ex. 10. Find the sum of 25 terms of the series,
2, 5, 8, 11, 14, &c.
ANSWER, 950.

Ex. 11. Find the sum of 36 terms of the series,
40, 38, 36, 34, &c.
ANSW. 180.

Ex. 12. Find the sum of 32 terms of the series,
1, $1\frac{1}{2}$, 2, $2\frac{1}{2}$, 3, &c.
ANSW. 280.

Ex. 13. The *sum* of an arithmetic series is 950, the *common difference* 3, and *number of terms* 25. What is the first term?
ANSW. 2.

Ex. 14. The *sum* of an arithmetic series is 165, the *first term* 3, and the *number of terms* 10. What is the *common difference?*

Answer, 3.

Ex. 15. The *sum* of an arithmetic series is 440, *first term* 3, and *common difference* 2. What are the *number of terms?*

Answ. 20.

Ex. 16. The *sum* of an arithmetic series is 54, *first term* 14, and *common difference* −2. What are the *number of terms?*

Answ. 9, or 6.

Ex. 17. A person bought 47 sheep, and gave 1 shilling for the *first* sheep, 3 for the *second*, 5 for the *third*, and so on. What did *all* the sheep cost him?

Answ. £.110. 9*s*.

Ex. 18. A person began the year by giving away a *farthing* the *first* day, a *halfpenny* the *second*, *three farthings* the *third*, and so on. What money had he disposed of in charity at the *end of the year?*

Answ. £.69. 11*s*. 6$\frac{3}{4}$*d*.

Ex. 19. *A* travels *uniformly* at the rate of 6 miles an hour, and sets off upon his journey 3 hours and 20 minutes before *B*; *B* follows him at the rate of 5 miles the *first* hour, 6 the *second*, 7 the *third*, and so on. In how many hours will *B* overtake *A?*

Answ. In 8 hours.

Ex. 20. There are a certain number of quantities in arithmetic progression, whose *first term* is 2, and whose *sum* is equal to 8 times their number; if 7 be added to the *third* term, and that sum be divided by the number of terms, the quotient will be equal to the *common difference*. What are the numbers?

Answ. 2, 5, 8, 11, 14.

XXXIII.

On Geometrical Progression.

112. Let S be the sum of the series a, ar, ar^2, ar^3, &c. (Art. 109), then

$$a+ar+ar^2+ar^3+\&c.\dots ar^{n-2}+ar^{n-1}\dots=S.$$

Multiply the equation by r, and it becomes

$$ar+ar^2+ar^3+\&c.\dots ar^{n-2}+ar^{n-1}+ar^n=rS.$$

Subtract the *upper* equation from the *lower*, and we have,

$$ar^n-a=rS-S, \text{ or } (r-1)S=ar^n-a;$$

$$\text{and therefore, } S=\frac{ar^n-a}{r-1}.$$

If r is a *proper fraction*, then r and *its powers* are less than 1. For the convenience of calculation, therefore, it is better in this case to transpose the equation into $S=\frac{a-ar^n}{1-r}$, by multiplying the numerator and denominator of the fraction $\frac{ar^n-a}{r-1}$ by -1.

113. When the number of terms is *large*, the summation of a series of this kind, and the general solution of the equation $S=\frac{ar^n-a}{r-1}$, or $\frac{a-ar^n}{1-r}$, can only be effected by means of *Logarithms.* (Vid. Chap. X.)

114. If l be the last term of a series of this kind, then $l=ar^{n-1}$, $\therefore rl=ar^n$; hence $S=\left(\frac{ar^n-a}{r-1}\right)=\frac{rl-a}{r-1}$. From this equation, therefore, if any three of the four quantities S, a, r, l be given, the fourth may be found. For $S=\frac{rl-a}{r-1}$; $a=rl-(r-1)S$; $r=\frac{S-a}{S-l}$, and $l=\frac{(r-1)S+a}{r}$.

Ex. 1.

EXAMPLE I.

Find the sum of the series 1, 3, 9, 27, &c. to 12 terms.

$$\left.\begin{array}{r}\text{Here } a=1,\\ r=3,\\ n=12;\end{array}\right\}\therefore S=\frac{ar^n-a}{r-1}=\frac{1\times3^{12}-1}{3-1}.$$

$$=\frac{81^3-1}{2}.$$

$$=\frac{531441-1}{2}=\frac{531440}{2}=265720.$$

Ex. 2.

Find the sum of ten terms of the series $1+\frac{2}{3}+\frac{4}{9}+\frac{8}{27}$, &c.

$$\left.\begin{array}{r}\text{Here } a=1\\ r=\frac{2}{3}\\ n=10\end{array}\right\}\therefore S=\frac{a-ar^n}{1-r}=\frac{1-\left(\frac{2}{3}\right)^{10}}{1-\frac{2}{3}}=\frac{\overline{1-\left(\frac{2}{3}\right)^{10}}\times3}{3-2}=\overline{1-\left(\frac{2}{3}\right)^{10}}\times3.$$

$$\text{Now } \left(\frac{2}{3}\right)^{10}=\frac{2^{10}}{3^{10}}=\frac{1024}{59049};$$

$$\therefore\ 1-\left(\frac{2}{3}\right)^{10}=1-\frac{1024}{59049}=\frac{58025}{59049},$$

$$\text{and } S=\frac{3+58025}{59049}=\frac{174075}{59049}.$$

Ex. 3. Find the sum of 1, 2, 4, 8, 16, &c. to 14 terms.

ANSWER, 16383.

Ex. 4. . . . . $1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}$, &c. to 8 terms.

ANSW. $\frac{3280}{2187}$.

XXXIV.

On the method of finding any number of Arithmetic or Geometric Means between Two Numbers.

115. Let l be the *last* term of an arithmetic series, whose *first* term is (a), *common difference* (b), and *number of terms* (n); then $l=a+(n-1)b$; $\therefore (n-1)b=l-a$, or $b=\frac{l-a}{n-1}$.

Now the number of *intermediate* terms between the first and the last is $n-2$; let $n-2=m$, then $n-1=m+1$. Hence $b=\frac{l-a}{m+1}$, which gives the following Rule for finding *any number* of arithmetic means between two numbers; "*Divide the difference of the two numbers by the given number of means increased by unity, and the quotient will be the common difference.*" Having the common difference, the *means* themselves will be known.

116. Let l be the last term of a *geometric* series, then $l=ar^{n-1}$, and $r^{n-1}=\frac{l}{a}$, $\therefore r=\sqrt[n-1]{\frac{l}{a}}$. The number of intermediate terms as before is $n-2$; let $n-2=m$, then $n-1=m+1$, and $r=\sqrt[m+1]{\frac{l}{a}}$, which gives the following rule for finding *any number* of geometric means between two numbers; viz. "*Divide one number by the other, and take that root of the quotient which is denoted by* $m+1$; *the result will be the common ratio.*" Having the *common ratio*, the means are found by continual multiplication.

EXAMPLE I.

Find *six* arithmetic means between 1 and 43.

$$\text{Here } \left.\begin{array}{r} l=43, \\ a=1, \\ m=6; \end{array}\right\} \therefore b=\frac{l-a}{m+1}=\frac{43-1}{6+1}=\frac{42}{7}=6.$$

By adding this common difference continually to the lesser number (1), we have 7, 13, 19, 25, 31, 37, for the *six* means required.

Ex. 2.

Find three geometric means between 2 and 32.

$$\text{Here } \left.\begin{array}{r} a=2, \\ l=32, \\ m=3; \end{array}\right\} \therefore r=\sqrt[m+1]{\frac{l}{a}}=\sqrt[4]{\frac{32}{2}}=\sqrt[4]{16}=2;$$

and the means required are, 4, 8, 16.

Ex. 3.

Find two geometric means between $\frac{16}{27}$ and 2.

$$\left.\begin{array}{r} \text{Here } a=\frac{16}{27}, \\ l=2, \\ m=2; \end{array}\right\} \begin{array}{l} \therefore r=\sqrt[m+1]{\frac{l}{a}}=\sqrt[3]{2\times\frac{27}{16}}=\sqrt[3]{\frac{27}{8}}=\frac{3}{2}. \\ \therefore \text{ the two means are } \frac{8}{9} \text{ and } \frac{4}{3}. \end{array}$$

Ex. 4. Find seven arithmetic means between 8 and 59.

ANSWER, 10, 17, 24, 31, 38, 45, 52.

Ex. 5. Find eight arithmetic means between 4 and 67.

Ex. 6. Find nine arithmetic means between 9 and 109.

Ex. 7. Find two geometric means between 4 and 256.

ANSW. 16 and 64.

Ex. 8. Find three geometric means between $\frac{1}{9}$ and 9.

ANSW. $\frac{1}{3}$, 1, 3.

If the number of geometric means required be *large*, the problem must be solved by means of *Logarithms*.

117. Let a, $a+b$, $a+2b$ be three quantities in arithmetic progression, then the sum of the first and last $=2a+2b=2(a+b)$; $\therefore a+b=$ *half* the sum of the first and last; hence "an arithmetic mean between any *two* quantities is found, by taking *half their sum.*" Again, let a, ar, ar^2 be any three quantities in *geometric* progression, then the *product* of the first and last $=a^2r^2=$ the *square* of the mean term, from which it appears that "a *geometric* mean between any two quantities is found by taking the *square root of their product*[a]." From

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[a] It may be proper here to observe, that quantities which are in *geometric progression* are also in *continued proportion*; for $a : ar :: ar : ar^2 :: ar^2 : ar^3 ::$ &c. The *differences* of quantities in geometric progression are also in continued

hence also it appears, that an *arithmetic* mean between any two numbers is greater than a *geometric* mean; for let the two numbers be x and y, then the *arithmetic* mean$=\frac{x+y}{2}$, and the geometric$=\sqrt{xy}$. Now $(x-y)^2$ is necessarily a *positive* quantity(b), and since it is equal to $x^2-2xy+y^2$, x^2+y^2 must be greater than $2xy$; add $2xy$ to each, then $x^2+2xy+y^2$ is greater than $4xy$, and consequently $x+y$ greater than $2\sqrt{xy}$, or $\frac{x+y}{2}$ greater than $\sqrt{xy}$.

XXXV.

On the solution of Equations relating to Numbers in Arithmetical or Geometrical Progression.

118. As a given number of terms of any arithmetic or geometric series may be expressed by means of *two unknown quantities*, it is not difficult to find a given number of quantities of this kind, which shall bear all such relations to each other as may be determined by *two equations*; of which the following are Examples.

Example I.

Find four numbers in arithmetical progression, such, that their *sum* shall be 56, and the *sum of their squares* 864.

Let x=the *second* of these four numbers,
and y=their *common difference*.

Then

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tinued proportion; for the successive differences of the terms of the series a, ar, ar^2, ar^3, ar^4, &c. are $ar-a$, ar^2-ar, ar^3-ar^2, &c. or $ar-a$, $(ar-a)r$, $(ar-a)r^2$, &c. which is a geometric progression whose *first term* is $ar-a$, and *common ratio* r.

(b) For $x-y$ is *positive* or *negative*, according as x is *greater* or *less* than y; but since the *square* of a quantity is necessarily *positive*, $(x-y)^2$ will be *positive*, whatever be the relative values of x and y.

Then the four numbers may be represented by $x-y$, x, $x+y$, $x+2y$.

Hence, by the question,

$$(x-y)+x+(x+y)+(x+2y)=4x+2y=56,$$

and $(x-y)^2+x^2+(x+y)^2+(x+2y)^2=4x^2+4xy+6y^2=864.$

From 1st equation, $2x+y=28$.

Square this equation, then $4x^2+4xy+y^2=784$ (A),

but $4x^2+4xy+6y^2=864$ (B).

Subtract (A) from (B), and we have $5y^2=80$,

or $y^2=16$, and $y=4$;

$$\therefore x=\frac{28-y}{2}=\frac{24}{2}=12.$$

Hence 8, 12, 16, 20 are the four numbers required.

Ex. 2.

The *sum* of three numbers in arithmetic progression is 9, and the *sum of their cubes* is 153. What are the numbers?

Let $x-y$, x, $x+y$, be the numbers.

Then $(x-y)+x+(x+y)=3x=9$,

$(x-y)^3+x^3+(x+y)^3=3x^3+6xy^2=153.$

From 1st equation, $x=\frac{9}{3}=3$;

$\therefore$ by *substitution*, in 2d equation, $81+18y^2=153$,

or $18y^2=153-81=72$;

$\therefore y^2=\frac{72}{18}=4$, and $y=2$.

Hence, the numbers are 1, 3, 5.

Ex. 3.

Find three numbers in geometric progression, such, that their *sum* shall be equal to 7, and the *sum of their squares* to 21.

Let x, y, z, be the numbers.

Then, by the question, $x+y+z=7$, 1st equation.

And $x^2+y^2+z^2=21$, 2d equation.

By Note (a), page 152, $x : y :: y : z$; $\therefore y^2 = xz$.

From 1st equation, $x + z = 7 - y$.

Square this equation, and $x^2 + 2xz + z^2 = 49 - 14y + y^2 \ (A)$;

but $2xz = \qquad 2y^2 \ (B)$.

Subtract (B) from (A), then $x^2 + z^2 = 49 - 14y - y^2$.

But, from *second* equation, $x^2 + z^2 = 21 - y^2$.

Hence, $49 - 14y - y^2 = 21 - y^2$,

or $49 - 14y = 21$;

$\therefore 14y = 49 - 21 = 28$.

$$\therefore y = \frac{28}{14} = 2.$$

Again, since $x + z = 7 - y = 7 - 2 = 5$,

we have $x^2 + 2xz + z^2 = 25$;

but $4xz = 16$, for $xz = y^2$;

$\therefore$ by subtraction, $x^2 - 2xz + z^2 = 25 - 16 = 9$,

and $x - z = 3$.

$$\left.\begin{aligned} \text{Hence, } x + z &= 5, \\ x - z &= 3; \end{aligned}\right\} \quad \begin{aligned} \therefore 2x &= 8, \text{ or } x = 4 \\ 2z &= 2, \text{ or } z = 1, \end{aligned}$$

and the three numbers are 1, 2, 4.

Ex. 4.

The *sum* of four numbers in geometric progression is 30, and the *last term divided by the sum of the mean terms* is $\frac{4}{3}$; What are the numbers?

$$\left.\begin{aligned} \text{Let } x &= \text{first term,} \\ y &= \text{the } \textit{common ratio}; \end{aligned}\right\} \text{then the numbers themselves will be } x,\ xy,\ xy^2,\ xy^3.$$

$$\left.\begin{aligned} \text{Hence, by the question, } x + xy + xy^2 + xy^3 &= 30, \text{ 1st equation,} \\ \text{and } \frac{xy^3}{xy + xy^2} &= \frac{4}{3}, \text{ 2d equation.} \end{aligned}\right\}$$

$$\left.\begin{array}{l} \text{From 1st} \\ \text{equation,} \end{array}\right\} x \times (1 + y + y^2 + y^3) = 30, \text{ or } x = \frac{30}{1 + y + y^2 + y^3} \ (A).$$

From

$$\left.\begin{array}{r}\text{From 2d}\\ \text{equation,}\end{array}\right\}\ \frac{xy\times y^2}{xy\times(1+y)}=\frac{4}{3},\ \text{or}\ \frac{y^2}{1+y}=\frac{4}{3}(B).$$

$$\left.\begin{array}{r}\text{By reduction of}\\ \text{equation }(B),\end{array}\right\}\ 3y^2=4+4y,$$

$$\text{or } y^2-\frac{4}{3}y=\frac{4}{3};$$

$$\therefore y^2-\frac{4}{3}y+\frac{4}{9}=\frac{4}{3}+\frac{4}{9}=\frac{16}{9},$$

$$\text{and } y-\frac{2}{3}=\frac{4}{3};\ \text{or } y=\frac{6}{3}=2.$$

Hence from equation (A), $x=\frac{30}{1+2+4+8}=\frac{30}{15}=2.$

The four numbers are therefore 2, 4, 8, 16.

Ex. 5.

There are three numbers in geometric progression, whose *product* is 64, and *sum of their cubes* 584; What are the numbers?

Let the numbers be x, xy, xy^2.

$$\left.\begin{array}{r}\text{Then, by the question, } x\times xy\times xy^2, \text{or } x^3y^3=64, \text{ 1st equation.}\\ \text{And } x^3+x^3y^3+x^3y^6=584, \text{ 2d equat}^{\text{n}}.\end{array}\right\}$$

From 1st equation, $y^3=\frac{64}{x^3}$, and $y^6=\frac{4096}{x^6}$.

$$\left.\begin{array}{r}\text{By substitution, in}\\ \text{2d equation,}\end{array}\right\}\ x^3+64+\frac{4096}{x^3}=584.$$

$$\begin{aligned}\text{Hence, } x^6+64x^3+4096&=584x^3,\\ \text{or } x^6-520x^3\qquad&=-4096.\end{aligned}$$

$$\left.\begin{array}{l}\text{Solve this equation by}\\ \text{Rule in Sect. XXIV.}\\ \text{page 109.}\end{array}\right\}\ \ldots \text{ and } x^3=8;\ \text{or } x=2.$$

$$\text{Now } y^3=\frac{64}{x^3}=\frac{64}{8}=8;\ \therefore y=2.$$

And the three numbers are 2, 4, 8.

Ex. 6. The sum of three numbers in arithmetic progression is 15; and the sum of the squares of the two extremes is 58. What are the numbers? ANSWER, 3, 5, 7.

Ex. 7. There are four numbers in arithmetic progression; the sum of the two extremes is 8, and the product of the means is 15. What are the numbers?

ANSWER, 1, 3, 5, 7.

Ex. 8. There are four numbers in arithmetic progression; the sum of the squares of the two means is 2, and the sum of the squares of the two extremes is 18. What are the numbers?

ANSW. $-3, -1, 1, 3$.

Ex. 9. There are three numbers in geometric progression, whose *sum* is 21, and *sum of their squares* 189. What are the numbers? ANSW. 3, 6, 12.

Ex. 10. There are three numbers in geometric progression; the sum of the *first* and *last* is 52, and the *square of the mean* is 100. What are the numbers?

ANSW. 2, 10, 50.

Ex. 11. There are three numbers in geometric progression, whose sum is 31, and the sum of the *first* and *last* is 26. What are the numbers? ANSW. 1, 5, 25.(a)

XXXVI.

On the Summation of an infinite Series of Fractions in Geometric Progression; and on the method of finding the value of Circulating Decimals.

119. The general expression for the sum of a geometric series whose common ratio (r) is a *fraction*, is (Art. 112) $S=\frac{a-ar^n}{1-r}$.

(a) Some curious Theorems relating to numbers in *Geometrical* Progression will be found in "*Elémens d'Algèbre* par L'Huiler," Vol. II. page 177...208. Ed. 1812. A great variety of questions, both in *Arithmetical* and *Geometrical* Progression, will also be found in Bland's "*Algebraical Problems*."

Suppose now n to be indefinitely *great*, then r^n (r being a proper fraction) will be indefinitely *small*[a], so that ar^n may be considered as *nothing* with respect to a in the numerator $a-ar^n$ of the fraction expressing the value of S; the *limit*, therefore, to which this value of S approaches, when the number of terms is infinite, is $\frac{a}{1-r}$.

EXAMPLE 1.

Find the sum of the series $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}$, &c. *ad infinitum*.

$$\left.\begin{array}{r}\text{Here } a=1,\\ r=\frac{1}{2};\end{array}\right\} \therefore S=\frac{a}{1-r}=\frac{1}{1-\frac{1}{2}}=\frac{2}{2-1}=2.$$

Ex. 2.

Find the value of $\frac{1}{5}+\frac{1}{25}+\frac{1}{125}+$&c. *ad infinitum*.

$$\left.\begin{array}{r}\text{Here } a=\frac{1}{5},\\ r=\frac{1}{5};\end{array}\right\} \therefore S=\frac{\frac{1}{5}}{1-\frac{1}{5}}=\frac{1}{5-1}=\frac{1}{4}.$$

Ex. 3.

Find the value of $\frac{3}{4}+\frac{1}{2}+\frac{1}{3}+\frac{2}{9}+\frac{4}{27}+$&c. *ad infinitum*.

$$\left.\begin{array}{r}\text{Here } a=\frac{3}{4},\\ r=\frac{2}{3};\end{array}\right\} \therefore S=\frac{\frac{3}{4}}{1-\frac{2}{3}}=\frac{3}{4-\frac{8}{3}}=\frac{9}{12-8}=\frac{9}{4}.$$

(a) When r is a proper fraction, it is evident that r^n *decreases* as n *increases*; let $r=\frac{1}{10}$ for instance, then $r^2=\frac{1}{100}$, $r^3=\frac{1}{1000}$, $r^4=\frac{1}{10000}$, &c. and when n is indefinitely great, the *denominator* of the fraction becomes so large with respect to the *numerator*, that the value of the fractional itself becomes less than any assignable quantity.

Ex. 4. Find value of $1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+$ &c. *ad infinitum.*

ANSWER, $\frac{3}{2}$.

Ex. 5. . . . . $1+\frac{3}{4}+\frac{9}{16}+\frac{27}{64}+$ &c. *ad infinitum.*

ANSW. 4.

Ex. 6. . . . . $\frac{2}{5}+\frac{4}{25}+\frac{8}{125}+$ &c. *ad infinitum.*

ANSW. $\frac{2}{3}$.

Ex. 7. . . . . $\frac{5}{3}+1+\frac{3}{5}+\frac{9}{25}+$ &c. *ad infinitum.*

ANSW. $4\frac{1}{6}$.

120. These operations furnish us with an expeditious method of finding the value of *circulating decimals,* the numbers composing which are geometric progressions, whose *common ratios* are $\frac{1}{10}$, $\frac{1}{100}$, $\frac{1}{1000}$, &c. according to the number of factors contained in the *repeating* decimal.

EXAMPLE 1.

Find the value of the circulating decimal .33333, &c. This decimal is represented by the geometric series $\frac{3}{10}+\frac{3}{100}+\frac{3}{1000}+$ &c. whose *first term* is $\frac{3}{10}$, and *common ratio* $\frac{1}{10}$.

$$\text{Hence } \left.\begin{aligned} a&=\frac{3}{10}, \\ r&=\frac{1}{10}, \end{aligned}\right\} \therefore S=\frac{a}{1-r}=\frac{\frac{3}{10}}{1-\frac{1}{10}}=\frac{3}{10-1}=\frac{3}{9}=\frac{1}{3}.$$

Ex. 2.

Ex. 2.

Find the value of .32323232, &c. *ad infinitum.*

$$\left.\begin{array}{l}\text{Here } a=\frac{32}{100},\\ \quad r=\frac{1}{100};\end{array}\right\}\therefore S=\frac{a}{1-r}=\frac{\frac{32}{100}}{1-\frac{1}{100}}=\frac{32}{100-1}=\frac{32}{99}.$$

Ex. 3.

Find the value of .713333, &c. *ad infinitum.*

The series of fractions representing the value of this decimal are $\frac{71}{100}+$ (geometric series) $\frac{3}{1000}+\frac{3}{10000}+$&c. $=\frac{71}{100}+S.$

$$\left.\begin{array}{l}\text{Here } a=\frac{3}{1000},\\ \quad r=\frac{1}{10};\end{array}\right\}\therefore S=\frac{\frac{3}{1000}}{1-\frac{1}{10}}=\frac{3}{1000-100}=\frac{3}{900}=\frac{1}{300}.$$

Hence the value of the decimal $=\left(\frac{71}{100}+S\right)\frac{71}{100}+\frac{1}{300}=\frac{214}{300}=\frac{107}{150}.$

Ex. 4.

Find the value of .81343434, &c. *ad infinitum.*

$$\left.\begin{array}{l}\text{Here } a=\frac{34}{10000},\\ \quad r=\frac{1}{100};\end{array}\right\}\therefore S=\frac{a}{1-r}=\frac{\frac{34}{10000}}{1-\frac{1}{100}}=\frac{34}{10000-100}=\frac{34}{9900}.$$

And the value of the decimal $=\frac{81}{100}+S=\frac{81}{100}+\frac{34}{9900}=\frac{8053}{9900}.$

Ex. 5. Find the value of .77777, &c. *ad infinitum.*

ANSWER, $\frac{7}{9}$.

Ex. 6. Find the value of .232323, &c. *ad infinitum.*

ANSWER, $\frac{23}{99}$.

Ex. 7. . . . . .83333, &c. *ad infinitum.*

ANSW. $\frac{5}{6}$.

Ex. 8. . . . . .7141414, &c. *ad infinitum.*

ANSW. $\frac{707}{990}$.

Ex. 9. . . . . .956666, &c. *ad infinitum.*

ANSW. $\frac{287}{300}$.

CHAP. VIII.

ON SURDS.

SURD quantities have already been defined in Art. 57. and may be expressed either by the *radical sign*, or by their *fractional indices* (as in Art. 68. I.); thus the *square root of* 2, the *cube root of* 3, the *n*th *root of* $a+b$, the *cube root of* $(a+x)^2$, &c. &c. may be expressed either by $\sqrt{2}$, $\sqrt[3]{3}$, $\sqrt[n]{a+b}$, $\sqrt[3]{(a+x)^2}$, &c. or by $2^{\frac{1}{2}}$, $3^{\frac{1}{3}}$, $(a+b)^{\frac{1}{n}}$, $(a+x)^{\frac{2}{3}}$, &c.

The precise value of these quantities cannot be ascertained; it can only be expressed by means of *decimals* or *series* which do not terminate; and in this sense they are called *irrational*, to distinguish them from all other quantities whatever, integral or fractional, whose values are determinate, and which are therefore denominated *rational*. Surds in their *radical* form, when properly reduced, are subject to all the ordinary Rules of Arithmetic.

XXXVII.

ON THE REDUCTION OF SURDS.

CASE I.

121. *A* RATIONAL *quantity may be reduced to the form of a surd, by raising it to the power denoted by the root of the surd.*

Ex. 1.

EXAMPLE I.

Reduce 3 to form of the *square root*, and it becomes $\sqrt{3^2}$ or $\sqrt{9}$.

Ex. 2.

Reduce $\frac{2}{3}$ . . . . . . *cube root*, ... $\sqrt[3]{\frac{2^3}{3^3}}$ or $\sqrt[3]{\frac{8}{27}}$.

Ex. 3.

Reduce $a+b$ . . . *square root*, ... $\sqrt{(a+b)^2}$

Ex. 4.

Reduce $4b^{\frac{2}{3}}$ . . . . . *cube root*, ... $\sqrt[3]{64b^2}$.

CASE II.

122. *Surds of different indices are reduced to equivalent ones with the same index, by bringing their fractional indices to a common denominator.*

Ex. 1. Reduce $a^{\frac{1}{2}}$ and $a^{\frac{1}{3}}$ to surds of the same index.

The fractions $\frac{1}{2}$, and $\frac{1}{3}$, reduced to a common denominator, are $\frac{3}{6}$ and $\frac{2}{6}$;

$$\left.\begin{array}{r} \therefore a^{\frac{1}{2}}=a^{\frac{3}{6}}=\sqrt[6]{a^3}, \\ \text{and } a^{\frac{1}{3}}=a^{\frac{2}{6}}=\sqrt[6]{a^2}; \end{array}\right\} \text{which are surds with the same index } \sqrt[6]{\ }.$$

Ex. 2. Reduce $3^{\frac{2}{3}}$ and $5^{\frac{1}{2}}$ to surds of the same index.

The fractions $\frac{2}{3}$ and $\frac{1}{2}$, reduced to a common denominator, are $\frac{4}{6}$ and $\frac{3}{6}$.

Now $3^{\frac{4}{6}}=\sqrt[6]{3^4}=\sqrt[6]{81}$; and $5^{\frac{3}{6}}=\sqrt[6]{5^3}=\sqrt[6]{125}$.

$$\left.\begin{array}{l} \text{Ex. 3. Reduce } a^{\frac{1}{3}} \text{ and } b^{\frac{1}{5}} \\ \text{Ex. 4. . . . . } c^{\frac{3}{4}} \text{ and } d^{\frac{1}{2}} \\ \text{Ex. 5. . . . . } 3\sqrt[3]{2} \,\&\, 2\sqrt{5} \\ \text{Ex. 6. . . . . } 4^{\frac{2}{3}} \text{ and } 15^{\frac{1}{2}} \end{array}\right\} \begin{array}{c} \text{to Surds,} \\ \text{with the} \\ \textit{same} \\ \text{index.} \end{array} \left\{\begin{array}{l} \text{ANS. } \sqrt[15]{a^5} \text{ and } \sqrt[15]{b^3}. \\ \text{. . . } \sqrt[4]{c^3} \text{ and } \sqrt[4]{d^2}. \\ \text{. . . } 3\sqrt[6]{4} \,\&\, 2\sqrt[6]{125}. \\ \text{. . . } \sqrt[6]{256} \,\&\, \sqrt[6]{3375}. \end{array}\right.$$

CASE III.

123. *Surds are reduced to their simplest form, by observing whether the quantity under the radical sign contains a power corresponding to the given surd root, and then extracting that root.*

EXAMPLES.

Ex. 1. $\sqrt{a^2b}=\sqrt{a^2}\times\sqrt{b}=a\sqrt{b}$.

Ex. 2. $\sqrt[m]{a^mx}=\sqrt[m]{a^m}\times\sqrt[m]{x}=a\sqrt[m]{x}$.

Ex. 3. $\sqrt{72}=\sqrt{36\times 2}=\sqrt{36}\times\sqrt{2}=6\sqrt{2}$.

Ex. 4. $\sqrt[3]{108}=\sqrt[3]{27\times 4}=\sqrt[3]{27}\times\sqrt[3]{4}=3\sqrt[3]{4}$.

Ex. 5. $\sqrt[3]{2a^3b^2+a^5bc}=\sqrt[3]{a^3\times(2b^2+a^2bc)}$.

$$=\sqrt[3]{a^3}\times\sqrt[3]{2b^2+a^2bc}.$$

$$=\quad a\sqrt[3]{2b^2+a^2bc}.$$

Ex. 6. Reduce $\sqrt{a^4bc}$ & $\sqrt{98a^2x}$	to their *simplest* form.	ANS. $a^2\sqrt{bc}$ & $7a\sqrt{2x}$.
Ex. 7 . . . . . . $\sqrt[3]{a^3+a^3b^6}$		. . . $a\sqrt[3]{1+b^6}$.
Ex. 8 . . . . . . $\sqrt{56}$ and $\sqrt[3]{72}$		. . . $2\sqrt{14}$ and $2\sqrt[3]{9}$.
Ex. 9 . . . . . . $\sqrt[4]{243}$ and $\sqrt[5]{96}$		. . . $3\sqrt[4]{3}$ and $2\sqrt[5]{3}$.

The quantity *without* the radical sign is called the *coefficient* of the surd; and it is evident, that this quantity may always be put *under* the radical sign, by raising it to the power denoted by the index of the surd.

Thus, $7a\sqrt{2x}=$ (by Case I.) $\sqrt{7a\times 7a}\times\sqrt{2x}$.

$$=\sqrt{49a^2}\times\sqrt{2x}=\sqrt{98a^2x}.$$

Also, $x\sqrt{2a-x}=\sqrt{x^2}\times\sqrt{2a-x}$.

$$=\sqrt{x^2\times(2a-x)}=\sqrt{2ax^2-x^3}.$$

CASE

CASE IV.

124. If the quantity under the radical sign be a *fraction*, it may be reduced to an *integral* form by the following process.

Multiply the numerator and denominator of the fraction by such a quantity as will make the denominator a complete power, corresponding to the root; then extract the root of the fraction whose numerator and denominator are complete powers, and take it from under the radical sign.

Ex. 1. $$\frac{c}{d}\times\sqrt{\frac{a^2}{b}}=\frac{c}{d}\times\sqrt{\frac{a^2b}{b^2}}.$$
$$=\frac{c}{d}\times\sqrt{\frac{a^2}{b^2}}\times\sqrt{b}.$$
$$=\frac{c}{d}\times\frac{a}{b}\times\sqrt{b}=\frac{ac}{bd}\sqrt{b}.$$

Ex. 2. $$\frac{3}{4}\times\sqrt{\frac{2}{7}}=\frac{3}{4}\times\sqrt{\frac{2\times7}{7\times7}}.$$
$$=\frac{3}{4}\sqrt{\frac{1}{49}\times14}.$$
$$=\frac{3}{4}\sqrt{\frac{1}{49}}\times\sqrt{14}.$$
$$=\frac{3}{4}\times\frac{1}{7}\times\sqrt{14}=\frac{3}{28}\sqrt{14}.$$

Ex. 3. $$\frac{1}{3}\sqrt[3]{\frac{16}{81}}=\frac{1}{3}\sqrt[3]{\frac{8\times2}{27\times3}}=\frac{1}{3}\times\frac{2}{3}\times\sqrt[3]{\frac{2}{3}}.$$
$$=\frac{2}{9}\times\sqrt[3]{\frac{2}{3}}.$$
$$=\frac{2}{9}\times\sqrt[3]{\frac{2\times3^2}{3^3}}.$$
$$=\frac{2}{9}\times\sqrt[3]{\frac{1}{27}\times18}.$$
$$=\frac{2}{9}\times\frac{1}{3}\times\sqrt[3]{18}=\frac{2}{27}\sqrt[3]{18}.$$

Ex. 4.

Ex. 4. Reduce $x\sqrt{\frac{b}{y}}$ and $a\sqrt[3]{\frac{c^2}{a}}$	to *integral* Surds in their simplest form.	Ans. $\frac{x}{y}\sqrt{by}$ and $\sqrt[3]{c^2a^2}$.
Ex. 5. . . . . $\sqrt{\frac{50}{147}}$ and $2\sqrt[3]{\frac{3}{4}}$		. . . . $\frac{5}{21}\sqrt{6}$ and $\sqrt[3]{6}$.
Ex. 6. . . . . $\sqrt[4]{\frac{1}{3}}$ and $\frac{2}{3}\sqrt{\frac{1}{2}}$		. . . . $\frac{1}{3}\sqrt[4]{27}$ and $\frac{1}{3}\sqrt{2}$.

XXXVIII.

On the application of the Fundamental Rules of Arithmetic to Surd Quantities.

125. *On the Addition and Subtraction of Surds.*

Rule. *Reduce them to their simplest form; and if the surd part be the same in both, then their sum or difference will be found by taking the sum or difference of their coefficients.*

Example 1.

Find the *sum* and *difference* of $\sqrt{16a^2x}$ and $\sqrt{4a^2x}$.

By Case 3. Sect. 80. $\sqrt{16a^2x}=4\overset{a}{\wedge}\sqrt{x}$,

and $\sqrt{4a^2x}=2a\sqrt{x}$;

$\therefore$ the *sum* $=4a\sqrt{x}+2a\sqrt{x}=(4a+2a)\times\sqrt{x}=6a\sqrt{x}$.

the *difference* $=4a\sqrt{x}-2a\sqrt{x}=(4a-2a)\times\sqrt{x}=2a\sqrt{x}$.

Ex. 2.

Find the *sum* and *difference* of $\sqrt[3]{192}$ and $\sqrt[3]{24}$.

By Case 3. Sect. 80. $\sqrt[3]{192}=\sqrt[3]{64\times3}=4a\sqrt[3]{3}$,

and $\sqrt[3]{24}=\sqrt[3]{8\times3}=2\sqrt[3]{3}$;

$\therefore\ \sqrt[3]{192}\pm\sqrt[3]{24}=(4\pm2)\times\sqrt[3]{3}=6\sqrt[3]{3}$ or $2\sqrt[3]{3}$.

Ex. 3.

Find the *sum* and *difference* of $\sqrt{\frac{8}{27}}$ and $\sqrt{\frac{1}{6}}$.

The two fractions $\frac{8}{27}$ and $\frac{1}{6}$, reduced to a *common denominator*, are $\frac{48}{162}$ and $\frac{27}{162}$.

$$\text{Now, } \sqrt{\frac{48}{162}} = \sqrt{\frac{16 \times 3}{81 \times 2}} = \frac{4}{9}\sqrt{\frac{3}{2}}.$$

$$\text{and } \sqrt{\frac{27}{162}} = \sqrt{\frac{9 \times 3}{81 \times 2}} = \frac{3}{9}\sqrt{\frac{3}{2}}.$$

$$\text{Hence } \sqrt{\frac{8}{27}} \pm \sqrt{\frac{1}{6}} = \left(\frac{4}{9} \pm \frac{3}{9}\right)\sqrt{\frac{3}{2}} = \frac{7}{9}\sqrt{\frac{3}{2}}, \text{ or } \frac{1}{9}\sqrt{\frac{3}{2}}.$$

If the surd part is not the *same* in the quantities to be added or subtracted from each other, it is evident that such addition or subtraction can only be performed by placing the signs $+$ or $-$ between them.

Ex. 4. Add $\sqrt{27\,a^4x}$ and $\sqrt{3\,a^4x}$ together . ANSW. $4a^2\sqrt{3x}$.

Ex. 5. . . . $\sqrt{128}$ and $\sqrt{72}$ . . . . $14\sqrt{2}$.

Ex. 6. . . . $\sqrt[3]{135}$ and $\sqrt[3]{40}$ . . . . $5\sqrt[3]{5}$.

Ex. 7. Subtract $3\sqrt{\frac{5}{27}}$ from $4\sqrt{\frac{3}{5}}$ . . . . $\frac{7}{15}\sqrt{15}$.

Ex. 8. . . . . . $\sqrt[3]{108}$ from $9\sqrt[3]{4}$ . . . . $6\sqrt[3]{4}$.

126. *On the Multiplication and Division of Surds.*

RULE.—*Reduce them to equivalent ones with the same index, and then multiply or divide both the rational and the irrational parts by each other respectively.*

EXAMPLE

EXAMPLE I.

Multiply $\sqrt{a}$ by $\sqrt[3]{b}$, or $a^{\frac{1}{2}}$ by $b^{\frac{1}{3}}$.

The fractions $\frac{1}{2}$ and $\frac{1}{3}$, reduced to *common denominators*, are $\frac{3}{6}$, and $\frac{2}{6}$;

$$\therefore\ a^{\frac{1}{2}}=a^{\frac{3}{6}}=\sqrt[6]{a^3};\ \text{and}\ b^{\frac{1}{3}}=b^{\frac{2}{6}}=\sqrt[6]{b^2}.$$

Hence $\sqrt{a}\times\sqrt[3]{b}=\sqrt[6]{a^3}\times\sqrt[6]{b^2}=\sqrt[6]{a^3b^2}$.

Ex. 2.

Multiply $5\sqrt{5}$ by $3\sqrt{8}$.

$$5\sqrt{5}\times3\sqrt{8}=15\sqrt{40}=15\sqrt{4\times10}.$$
$$=15\times2\times\sqrt{10}=30\sqrt{10}.$$

Ex. 3.

Multiply $2\sqrt{3}$ by $3\sqrt[3]{4}$.

By *reduction*, $2\sqrt{3}=2\times3^{\frac{3}{6}}=2\times\sqrt[6]{3^3}=2\sqrt[6]{27}$,

and $3\sqrt[3]{4}=3\times4^{\frac{2}{6}}=3\times\sqrt[6]{4^2}=3\sqrt[6]{16}$.

Hence $2\sqrt{3}\times3\sqrt[3]{4}=2\sqrt[6]{27}\times3\sqrt[6]{16}=6\sqrt[6]{432}$.

Ex. 4.

Divide $2\sqrt[3]{bc}$ by $3\sqrt{ac}$.

Now $2\sqrt[3]{bc}=2\times(bc)^{\frac{2}{6}}=2\sqrt[6]{b^2c^2}$,

and $3\sqrt{ac}=3\times(ac)^{\frac{3}{6}}=3\sqrt[6]{a^3c^3}$;

$$\therefore\ \frac{2\sqrt[3]{bc}}{3\sqrt{ac}}=\frac{2}{3}\times\sqrt[6]{\frac{b^2c^2}{a^3c^3}}=\frac{2}{3}\sqrt[6]{\frac{b^2}{a^3c}}.$$

Ex. 5.

Divide $10\sqrt[3]{108}$ by $5\sqrt[3]{4}$.

Now $10\sqrt[3]{108}=10\sqrt[3]{27\times4}=10\times3\times\sqrt[3]{4}=30\sqrt[3]{4}$;

$$\therefore\ \frac{10\sqrt[3]{108}}{5\sqrt[3]{4}}=\frac{30\sqrt[3]{4}}{5\sqrt[3]{4}}=6;\quad\text{or,}\quad\frac{10\sqrt[3]{108}}{5\sqrt[3]{4}}=2\sqrt[3]{27}.$$

$=2\times3=6.$

Ex. 6. Multiply $\sqrt[3]{15}$ by $\sqrt{10}$ . . . ANSWER, $\sqrt[6]{225000}$.

Ex. 7. . . . . . $\frac{1}{2}\sqrt[3]{6}$ by $\frac{2}{3}\sqrt[3]{18}$ . . . . $\sqrt[3]{4}$.

Ex. 8. Divide $10\sqrt{27}$ by $2\sqrt{3}$ . . . . 15.

Ex. 9. $10\sqrt[3]{108}$ by $5\sqrt[3]{84}$ . . . . $\frac{2}{7}\sqrt[3]{441}$.

127. *On the Involution and Evolution of Surds.*

RULE. *Raise the rational part to the power or root required, and then multiply the fractional index of the surd part by the index of that power or root.*

Ex. 1. The *square* of $\sqrt[3]{a}=a^{\frac{1}{3}\times 2}=a^{\frac{2}{3}}=\sqrt[3]{a^2}$.

Ex. 2. . . . . *Cube* of $\sqrt[5]{b^2}=b^{\frac{2}{5}\times 3}=b^{\frac{6}{5}}=\sqrt[5]{b^6}$.

Ex. 3. *4th power* of $2\sqrt[3]{2}=16\times 2^{\frac{1}{3}\times 4}=16\times 2^{\frac{4}{3}}=16\sqrt[3]{16}$
[$=32\sqrt[3]{2}$.

Ex. 4. *Square root* of $a^{\frac{1}{3}}b^{\frac{1}{2}}=a^{\frac{1}{3}\times\frac{1}{2}}b^{\frac{1}{2}\times\frac{1}{2}}=a^{\frac{1}{6}}b^{\frac{1}{4}}$.

Ex. 5. *Cube root* of $\frac{1}{8}\sqrt{2}=\frac{1}{2}\times 2^{\frac{1}{2}\times\frac{1}{3}}=\frac{1}{2}\times 2^{\frac{1}{6}}=\frac{1}{2}\sqrt[6]{2}$.

Ex. 6. Square $\sqrt[3]{5}$ . . . . ANSWER, $\sqrt[3]{25}$.

Ex. 7. Cube $\frac{1}{2}\sqrt{3}$ . . . . $\frac{1}{8}\sqrt{27}$.

Ex. 8. Find fourth power of $\frac{1}{6}\sqrt{6}$ . . $\frac{1}{36}$.

Ex. 9. . . . square root of $9\sqrt[3]{3}$ . . . $3\sqrt[6]{3}$.

Ex. 10. . . . fourth root of $\frac{16}{81}\sqrt[3]{a^2}$ . . $\frac{2}{3}\sqrt[6]{a}$,

Ex. 11. . . . fifth root of $\frac{1}{32}\times\left(\frac{b^2}{a}\right)^3$ . . $\frac{\sqrt[5]{b^6}}{2\sqrt[5]{a^3}}$.

128. From the preceding rules we easily deduce the method of converting fractions whose denominators are *surd* quantities, into others whose denominators shall be *rational*. Thus, let both the numerator and denominator of the fraction $\frac{a}{\sqrt{x}}$ be multiplied by $\sqrt{x}$; and it becomes $\frac{a\sqrt{x}}{x}$; and by multiplying the numerator and denominator of the fraction $\frac{b}{\sqrt[3]{a+x}}$ by $\sqrt[3]{(a+x)^2}$ or $(a+x)^{\frac{2}{3}}$, it becomes $\frac{b(a+x)^{\frac{2}{3}}}{\sqrt[3]{(a+x)^3}}=\frac{b(a+x)^{\frac{2}{3}}}{a+x}$. Or in general, if both the numerator and denominator of a fraction of the form $\frac{a}{\sqrt[n]{x}}$ be multiplied by $\sqrt[n]{x^{n-1}}$, it becomes $\frac{a\sqrt[n]{x^{n-1}}}{x}$, a fraction whose denominator is a *rational* quantity.

XXXIX.

On the method of finding Multipliers which shall render Binomial Surd Quantities Rational.

129. *Compound* surd quantities are such as consist of two or more terms, some or all of which are *irrational;* and if a quantity of this kind consist only of *two* terms, it is called a *binomial* surd. The rule for finding a multiplier which shall render a binomial surd quantity *rational*, is derived from observing the quotient which arises from the actual division of the numerator by the denominator of the following fractions. Thus,

I. $\frac{x^n-y^n}{x-y}=x^{n-1}+x^{n-2}y+x^{n-3}y^2+\&c\ldots+y^{n-1}$ to n terms, whether n be *even* or *odd*.

II. $\frac{x^n-y^n}{x+y}=x^{n-1}-x^{n-2}y+x^{n-3}y^2-\&c\ldots-y^{n-1}$ to n terms, when n is an *even* number.

III. x^n+

III. $\frac{x^n+y^n}{x+y}=x^{n-1}-x^{n-2}y+x^{n-3}y^2-\&c.\ldots+y^{n-1}$ to n terms, when n is an *odd* number[a].

130. Now let $x^n=a$, $y^n=b$, then $x=\sqrt[n]{a}$, $y=\sqrt[n]{b}$, and these fractions severally become $\frac{a-b}{\sqrt[n]{a}-\sqrt[n]{b}}$, $\frac{a-b}{\sqrt[n]{a}+\sqrt[n]{b}}$ and $\frac{a+b}{\sqrt[n]{a}+\sqrt[n]{b}}$; and by the application of the rules in Art. 68, 126, 127, we have $x^{n-1}=\sqrt[n]{a^{n-1}}$; $x^{n-2}=\sqrt[n]{a^{n-2}}$; $x^{n-3}=\sqrt[n]{a^{n-3}}$, &c.; also, $y^2=\sqrt[n]{b^2}$; $y^3=\sqrt[n]{b^3}$; &c.; hence, $x^{n-2}y=\sqrt[n]{a^{n-2}}\times\sqrt[n]{b}=\sqrt[n]{a^{n-2}b}$; $x^{n-3}y^2=\sqrt[n]{a^{n-3}}\times\sqrt[n]{b^2}=\sqrt[n]{a^{n-3}b^2}$; &c. By substituting these values of x^{n-1}, $x^{n-2}y$, $x^{n-3}y^2$, &c. in the several quotients, we have

$$\frac{a-b}{\sqrt[n]{a}-\sqrt[n]{b}}=\sqrt[n]{a^{n-1}}+\sqrt[n]{a^{n-2}b}+\sqrt[n]{a^{n-3}b^2}+\&c.\ldots\ldots+\sqrt[n]{b^{n-1}}$$

to n terms; where n may be any whole number whatever.

And

$$\frac{a+b}{\sqrt[n]{a}+\sqrt[n]{b}}=\sqrt[n]{a^{n-1}}-\sqrt[n]{a^{n-2}b}+\sqrt[n]{a^{n-3}b^2}+\&c.\ldots\ldots\pm\sqrt[n]{b^{n-1}}$$

to n terms; where the terms b and $\sqrt[n]{b^{n-1}}$ have the sign $+$, when n is an *odd* number; and the sign $-$, when n is an *even* number.

131. Since

(a) For I. $\frac{x^2-y^2}{x-y}=x+y$; $\frac{x^3-y^3}{x-y}=x^2+xy+y^2$; $\frac{x^4-y^4}{x-y}=x^3+x^2y+xy^2+y^3$; &c.

II. $\frac{x^2-y^2}{x+y}=x-y$; $\frac{x^4-y^4}{x+y}=x^3-x^2y+xy^2-y^3$; &c.

III. $\frac{x+y}{x+y}=1$; $\frac{x^3+y^3}{x+y}=x^2-xy+y^2$; $\frac{x^5+y^5}{x+y}=x^4-x^3y+x^2y^2-xy^3+y^4$; &c.

131. Since the *divisor* multiplied by the *quotient* gives the *dividend,* it appears from the foregoing operations that "if "a binomial surd of the form $\sqrt[n]{a}-\sqrt[n]{b}$ be multiplied by "$\sqrt[n]{a^{n-1}}+\sqrt[n]{a^{n-2}b}+\sqrt[n]{a^{n-3}b^2}+$ &c....$+\sqrt[n]{b^{n-1}}$ (n being any "whole number whatever), the product will be $a-b$, a *rational* "quantity; and if a binomial surd of the form $\sqrt[n]{a}+\sqrt[n]{b}$ be "multiplied by $\sqrt[n]{a^{n-1}}-\sqrt[n]{a^{n-2}b}+\sqrt[n]{a^{n-3}b^2}-$ &c....$\pm\sqrt[n]{b^{n-1}}$, "the product will be $a+b$ or $a-b$, according as the index n is "an *odd* or an *even* number." The great use of this rule is, "to convert fractions having *surd* denominators, into others "which shall have *rational* ones;" of which the following are Examples.

EXAM. I.

Reduce $\dfrac{x}{a-\sqrt{x}}$ and $\dfrac{\sqrt{6}}{\sqrt{8}+\sqrt{3}}$ to fractions having rational denominators.

Since "the *sum* into the *difference* of two quantities gives the *difference of their squares,*" it is evident that these fractions may be reduced to others having *rational* denominators, by multiplying their numerators and denominators by $a+\sqrt{x}$ and $\sqrt{8}-\sqrt{3}$ respectively, without the formal application of the rule.

$$\left.\begin{array}{l}\text{Thus } x\times(a+\sqrt{x})=ax+x\sqrt{x}\\ \text{and } (a-\sqrt{x})(a+\sqrt{x})=a^2-x\end{array}\right\}\begin{array}{l}\text{by which means the fraction}\\ \text{is reduced to } \dfrac{ax+x\sqrt{x}}{a^2-x}.\end{array}$$

Again $\sqrt{6}(\sqrt{8}-\sqrt{3})=\sqrt{48}-\sqrt{18}=$ (Art. 123) $4\sqrt{3}-3\sqrt{2}$,
and $(\sqrt{8}+\sqrt{3})(\sqrt{8}-\sqrt{3})=8-3=5$;

and the fraction is reduced to $\dfrac{4\sqrt{3}-3\sqrt{2}}{5}$.

Ex. 2.

Reduce $\dfrac{2}{\sqrt[3]{3}-\sqrt[3]{2}}$ to a fraction with a rational denominator.

To find the multiplier which shall make $\sqrt[3]{3}-\sqrt[3]{2}$ *rational,* we have $n=3$, $a=3$, $b=2$; $\therefore \sqrt[n]{a^{n-1}}+\sqrt[n]{a^{n-2}b}+\sqrt[n]{b^{n-1}}$(*)$=\sqrt[3]{9}+\sqrt[3]{6}+\sqrt[3]{4}$.

Now $2(\sqrt[3]{9}+\sqrt[3]{6}+\sqrt[3]{4})=2\sqrt[3]{9}+2\sqrt[3]{6}+2\sqrt[3]{4}$,

and $(\sqrt[3]{3}-\sqrt[3]{2})(\sqrt[3]{9}+\sqrt[3]{6}+\sqrt[3]{4})=(a-b)3-2=1$;

$\therefore$ the *denominator* is 1, and the fraction is reduced to $2\sqrt[3]{9}+2\sqrt[3]{6}+2\sqrt[3]{4}$.

Ex. 3.

Reduce $\frac{c}{\sqrt[3]{x}+\sqrt[3]{y}}$ to a fraction with a rational denominator.

Here $n=3$, $a=x$, $b=y$, the sign of $\sqrt[n]{b}$ is $+$, and n an *odd* number; $\therefore$ the multiplier is $\sqrt[n]{a^{n-1}}-\sqrt[n]{a^{n-2}b}+\sqrt[n]{b^{n-1}}=\sqrt[3]{x^2}-\sqrt[3]{xy}+\sqrt[3]{y^2}$; Hence the fraction required is

$$\left(\frac{c}{\sqrt[3]{x}+\sqrt[3]{y}}\right)\cdot\left(\frac{\sqrt[3]{x^2}-\sqrt[3]{xy}+\sqrt[3]{y^2}}{\sqrt[3]{x^2}-\sqrt[3]{xy}+\sqrt[3]{y^2}}\right)=\frac{c}{x+y}\left(\sqrt[3]{x^2}-\sqrt[3]{xy}+\sqrt[3]{y^2}\right).$$

Ex. 4.

Reduce $\frac{3}{\sqrt[4]{5}+\sqrt[4]{3}}$ to a fraction with a rational denominator.

Here $n=4$, $a=5$, $b=3$, the sign of $\sqrt[n]{b}$ is $+$, and n an *even* number, $\therefore$ the multiplier is $\sqrt[n]{a^{n-1}}-\sqrt[n]{a^{n-2}b}+\sqrt[n]{a^{n-3}b^2}-\sqrt[n]{b^{n-1}}=\sqrt[4]{125}-\sqrt[4]{75}+\sqrt[4]{45}-\sqrt[4]{27}$. Hence the fraction required is

$$\left(\frac{3}{\sqrt[4]{5}+\sqrt[4]{3}}\right)\left(\frac{\sqrt[4]{125}-\sqrt[4]{75}+\sqrt[4]{45}-\sqrt[4]{27}}{\sqrt[4]{125}-\sqrt[4]{75}+\sqrt[4]{45}-\sqrt[4]{27}}\right)$$

$$=\frac{3}{2}(\sqrt[4]{125}-\sqrt[4]{75}+\sqrt[4]{45}-\sqrt[4]{27}).$$

XL.

(*) The *number* of terms of the general series to be taken, is always equal to n; in the present instance, therefore, the number to be taken is 3; and so in all other cases; recollecting that the *last* terms is always $\sqrt[n]{b^{n-1}}$.

XL.

On the method of extracting the Square Root of Binomial Surds.

132. Let $\sqrt{x}$ and $\sqrt{y}$ be two *irreducible* quadratic surds, then their product $\sqrt{xy}$ will also be an irreducible quadratic surd. For, if possible, let $\sqrt{xy}=z$ where z is some *rational* number; then $xy=z^2$, from which it would follow that the product of any two numbers must *necessarily* be a *square* number; but that only happens when the two numbers are the *first* and *third* of three numbers which are in continued proportion.

133. Next, let $\sqrt{x}+\sqrt{y}$ be a binomial quadratic surd, each of whose terms is an irreducible surd. If this binomial be squared, the result is $x+y+2\sqrt{xy}$, a quantity, one part of which is *rational* and the other (by Art. 132.) an *irreducible surd*. Let the *rational* part $=a$, and the *irrational* $=\sqrt{b}$, then it appears that every binomial quadratic surd whose square root can be exhibited under the form $\sqrt{x}+\sqrt{y}$, must be of the form $a+\sqrt{b}$; a being a rational quantity, and $\sqrt{b}$ an irreducible surd.

134. The square root of a rational quantity cannot be partly *rational* and partly *irrational*. For, if possible, let $\sqrt{x}=a\pm\sqrt{b}$, where $\sqrt{b}$ is an irreducible surd; then $x=a^2+b\pm 2a\sqrt{b}$, and $\sqrt{b}=\frac{x-a^2-b}{2a}$, a *rational* quantity. But, by the supposition, $\sqrt{b}$ is an *irreducible surd;* hence $\sqrt{x}$ cannot be expressed under the form $a+\sqrt{b}$. In the same manner it may be proved, that the square root of a rational quantity cannot be equal to the *sum* or *difference* of two other quadratic surds. For, if possible, let $\sqrt{x}=\sqrt{a}\pm\sqrt{b}$, then $x=a+b\pm 2\sqrt{ab}$, and $\sqrt{ab}=\frac{x-a-b}{\pm 2}$, which is impossible by Art. 132.

135. In any equation, $x+\sqrt{y}=a+\sqrt{b}$, consisting of rational quantities and irreducible surds, the *rational* parts on each side are equal, and also the *irrational*. For if x be not equal to a, let $x=a\pm m$, then $a\pm m+\sqrt{y}=a+\sqrt{b}$, or $\pm m+\sqrt{y}$ $=\sqrt{b}$, i.e. $\sqrt{b}$ is partly *rational* and partly *irrational*, which

has already been proved to be impossible. In a similar manner it may be shewn, that in any equation $\sqrt{x}+\sqrt{y}=\sqrt{a}+\sqrt{b}$ (where $\sqrt{x}$ and $\sqrt{a}$, $\sqrt{y}$ and $\sqrt{b}$, respectively involve the same irreducible quadratic surd) $\sqrt{x}$ must be equal to $\sqrt{a}$, and $\sqrt{y}$ to $\sqrt{b}$. For if $\sqrt{x}$ be not equal to $\sqrt{a}$, let it be equal to $\sqrt{m}$; then $\sqrt{m}+\sqrt{y}=\sqrt{a}+\sqrt{b}$, or $\sqrt{m}=\sqrt{a}+\sqrt{b}-\sqrt{y}$; but since $\sqrt{b}$ and $\sqrt{y}$ contain the same irreducible surd, $\sqrt{b}-\sqrt{y}$ may be expressed under the form $p\sqrt{y}$, $\therefore \sqrt{m}=\sqrt{a}+p\sqrt{y}$, which is impossible by Art. 134.

136. To find the square root of the binomial quadratic surd $a+\sqrt{b}$. Assume $\sqrt{x}+\sqrt{y}=\sqrt{a+\sqrt{b}}$, then $x+y+2\sqrt{xy}=a+\sqrt{b}$; $\therefore$ (by Art. 134.) $x+y=a$, and $2\sqrt{xy}=\sqrt{b}$; hence $x^2+2xy+y^2=a^2$ (A), and $4xy=b$ (B); subtract (B) from (A), then $x^2-2xy+y^2=a^2-b$, and $x-y=\sqrt{a^2-b}$; we have therefore,

$$\left.\begin{array}{l} x+y=a \\ x-y=\sqrt{a^2-b} \end{array}\right\} \therefore \begin{array}{l} 2x=a+\sqrt{a^2-b}, \text{ and } x=\tfrac{1}{2}a+\tfrac{1}{2}\sqrt{a^2-b}. \\ 2y=a-\sqrt{a^2-b}, \text{ and } y=\tfrac{1}{2}a-\tfrac{1}{2}\sqrt{a^2-b}. \end{array}$$

Hence $\sqrt{x}+\sqrt{y}=\sqrt{\tfrac{1}{2}a+\tfrac{1}{2}\sqrt{a^2-b}}+\sqrt{\tfrac{1}{2}a-\tfrac{1}{2}\sqrt{a^2-b}}$, an expression which can evidently be of the form $\sqrt{x}+\sqrt{y}$, only when $\sqrt{a^2-b}$ is a *rational* quantity. The square root of the binomial surd quantity $a+\sqrt{b}$ can therefore be exhibited under the form $\sqrt{x}+\sqrt{y}$ only when a^2-b is a *square number*. By a similar process it might be shewn that the square root of $a-\sqrt{b}$ is $\sqrt{\tfrac{1}{2}a+\tfrac{1}{2}\sqrt{a^2-b}}-\sqrt{\tfrac{1}{2}a-\tfrac{1}{2}\sqrt{a^2-b}}$, subject to the same limitation.

EXAMPLE 1.

What is the square root of $19+8\sqrt{3}$?

$$\left.\begin{array}{l} \text{Here } a=19 \\ \sqrt{b}=8\sqrt{3} \end{array}\right\} \therefore a^2-b=361-192=169, \text{ and } \sqrt{a^2-b}=13.$$

$$\text{Hence } \sqrt{\tfrac{1}{2}a+\tfrac{1}{2}\sqrt{a^2-b}}+\sqrt{\tfrac{1}{2}a-\tfrac{1}{2}\sqrt{a^2-b}}=\sqrt{\frac{19}{2}+\frac{13}{2}}$$

$$+\sqrt{\frac{19}{2}-\frac{13}{2}}=\sqrt{16}+\sqrt{3}=4+\sqrt{3}.$$

Ex. 2.

Ex. 2.

Find the square root of $12-\sqrt{140}$.

Here $\left.\begin{array}{l} a=12 \\ \sqrt{b}=\sqrt{140} \end{array}\right\}$ $\therefore a^2-b=144-140=4$, and $\sqrt{a^2-b}=2$.

Hence $\sqrt{\tfrac{1}{2}a+\tfrac{1}{2}\sqrt{a^2-b}}-\sqrt{\tfrac{1}{2}a-\tfrac{1}{2}\sqrt{a^2-b}}=\sqrt{6+1}-\sqrt{6-1}=\sqrt{7}-\sqrt{5}$.

Ex. 3.

Find the square root of $31+12\sqrt{-5}$.

Here $\left.\begin{array}{l} a=31 \\ \sqrt{b}=12\sqrt{-5} \\ \text{or } b=-720 \end{array}\right\}$ $\therefore a^2-b=961+720=1681$, and $\sqrt{a^2-b}=41$.

Hence $\sqrt{\tfrac{1}{2}a+\tfrac{1}{2}\sqrt{a^2-b}}+\sqrt{\tfrac{1}{2}a-\tfrac{1}{2}\sqrt{a^2-b}}=\sqrt{\frac{31}{2}+\frac{41}{2}}+\sqrt{\frac{31}{2}-\frac{41}{2}}=6+\sqrt{-5}$.

Ex. 4. Find the square root of $7+4\sqrt{3}$ . . ANSW. $2+\sqrt{3}$.

Ex. 5. . . . . $7-2\sqrt{10}$ . . . $\sqrt{5}-\sqrt{2}$.

Ex. 6. . . . . $18-10\sqrt{-7}$ . . $5-\sqrt{-7}$.

CHAP. IX.

ON THE INVESTIGATION OF THE BINOMIAL THEOREM; AND ON THE EXPANSION AND REVERSION OF SERIES.

PREVIOUS to the investigation of the Binomial Theorem, it is necessary to explain the manner of finding the number of *permutations* or *combinations* which can take place amongst any set of quantities.

XLI.

On Permutations and Combinations.

137. By *Permutations* are meant the number of *changes* which any quantities a, b, c, d, e, &c. may undergo with respect to their order, when taken *two and two* together, *three and three*, &c. &c. Thus ab, ac, ad, ba, bc, bd, ca, cb, cd, da, db, dc, are the different permutations of the *four* quantities a, b, c, d, when taken *two and two* together; abc, acb, bac, bca, cab, cba, of the *three* quantities a, b, c, when taken *three and three* together; &c. &c.

138. By *Combinations* are meant the number of *collections* which may be formed out of the quantities a, b, c, d, e, &c. taken *two and two* together, *three and three* together, &c. &c.

without having regard to the *order* in which the quantities are arranged in each collection. Thus ab, ac, ad, bc, bd, cd, are the *combinations* which can be formed out of the *four* quantities a, b, c, d, taken *two and two* together; abc, abd, acd, bcd, the combinations which may be formed out of the same quantities, when taken *three and three* together; &c. &c.

139. Let there be n quantities, a, b, c, d, e, &c., then, by Art. 137, it appears that there will be $(n-1)$ permutations in which a stands first; for the same reason there will be $(n-1)$ permutations in which b stands first; and so of c, d, e, &c. Hence there will be n times $(n-1)$ permutations of the form ab, ac, ad, ae, &c.; ba, bc, bd, be, &c.; ca, cb, cd, ce, &c.; i.e. "*the number of permutations of n things taken two and two is $n(n-1)$.*"

140. If these n quantities be taken *three and three* together, then there will be $n(n-1)$ $(n-2)$ *permutations.* For if $(n-1)$ be substituted for n in the last article, then the number of permutations of $n-1$ things taken *two and two* together will be $(n-1)(n-2)$; hence the number of permutations of $b,c,d,e,$&c. taken *two and two* together, are $(n-1)(n-2)$, and consequently there are $(n-1)(n-2)$ permutations of the quantities a, b, c, d, e, &c. taken *three and three* together, in which a may stand first; for the same reason there are $(n-1)(n-2)$ permutations in which b may stand first; and so of c, d, e, &c. The numbers of the permutations of this kind will therefore amount to $n(n-1)(n-2)$.

141. In general therefore it appears, that if the number of quantities be n, and they are taken m *and* m together, the number of permutations will be $n(n-1)$ $(n-2)$ &c........ $(n-m+1)$; and if $m=n$, i.e. if the permutations respect all the quantities at once, then (since $m-n=0$) the *number* of them will be $n(n-1)(n-2)$ &c.....2.1. Thus, the number of

permutations which might be formed from the letters composing the word "*virtue*" are $6\times5\times4\times3\times2\times1=720$.

142. But if the *same* letter should occur any number of times, then it is evident that we must *divide* the whole number of permutations, by the number of permutations which would have arisen if *different* letters had occurred instead of the repetition of the same letter. Thus if the same letter should occur *twice*, then we must divide by 2×1; if it should occur *thrice*, we must divide by $3\times2\times1$; if p times, by $1.2.3\ldots p$; and so for any other letter which may occur more than once. Hence the general expression for the number of permutations of n things, of which there are p of *one* kind; r of *another*; q of *another*; &c. &c. is $\frac{n(n-1)(n-2)n-3\ldots.2.1}{1.2.3..p\times1.2.3..r\times1.2.3..q}$. Thus the permutations which may be formed by the letters composing the word "*easiness*" (since *s* occurs *thrice*, *e twice*) are $\frac{8.7.6.5.4.3.2.1}{1.2.3.\times1.2}=3360$.

143. From the expression (in Art. 141) for finding the number of *permutations* of n things taken m *and* m together, we immediately deduce the theorem for finding the number of *combinations* of n things taken in the same manner. For the permutations of n things taken *two and two* together being $n(n-1)$, and as each *combination* admits of as many *permutations* as may be made by *two* things (which is 2×1), the number of *combinations* must be equal to the number of *permutations* divided by 2; i.e. the number of *combinations* of n things taken *two and two* together is $\frac{n(n-1)}{2}$. For the same reason, the *combinations* of n things, taken *three and three* together, must be equal to $\frac{n(n-1)(n-2)}{1.2.3}$; and in general, the combinations of n things taken m *and* m together must be equal to $\frac{n(n-1)(n-2)\ldots.(n-m+1)}{1.2.3\ldots.m}$.

XLII.

XLII.

On the Investigation of the Binomial Theorem.

144. In Chap. III. it was shewn, that, by raising the successive powers of $(a+b)$ in the ordinary way of multiplication, the nth power of $a+b$ might be exhibited under the form $a^n+na^{n-1}b+\frac{n(n-1)}{1.2}a^{n-2}b^2+\frac{n(n-1)(n-2)}{1.2.3}a^{n-3}b^3+$ &c. For the investigation of this Theorem, let us suppose the quantities, $(x+a)$, $(x+b)$, $(x+c)$, $(x+d)$, &c. multiplied together, as in the following Table.

$$\begin{array}{l}
x+a \\
x+b \\
\hline
x^2+ax \\
\quad +bx+ab \\
\hline
x^2+(a+b)x+ab=(x+a)(x+b) \\
x+c \\
\hline
x^3+(a+b)x^2+ab.x \\
\qquad +c.x^2+(ac+bc)x+abc \\
\hline
x^3+(a+b+c)x^2+(ab+ac+bc)x+abc=(x+a)(x+b)(x+c) \\
x+d \\
\hline
x^4+(a+b+c)x^3+(ab+ac+bc)x^2+abc.x \\
\qquad\qquad +d.x^3+(ad+bd+cd)x^2+(abd+acd+bcd)x+abcd \\
\hline
x^4+(a+b+c+d)x^3+(ab+ac+ad+bc+bd+cd)x^2+(abc+abd+acd+bcd)x \\
+abcd=(x+a)(x+b)(x+c)(x+d) \\
\&c. \qquad \&c. \qquad \&c.
\end{array}$$

145. By the mere inspection of this Table, it appears that if there had been n quantities a, b, c, d, e, &c. then the *first* term of the product would be x^n; the *second* Px^{n-1}, where $P=(a+b+c+d+e)$, &c.; the third Qx^{n-2}, where $Q=$

$Q=(ab+ac+ad+ae+be+bd+$&c.); the *fourth* Rx^{n-3}, where $R=(abc+abd+acd+ace+bcd+bce+$&c.); the *m*th *term* Sx^{n-m+1}, where S=*sum of the combinations of any* $m-1$ *quantities*; &c. &c.; and the product itself would be expressed by $x^n+Px^{n-1}+Qx^{n-2}+Rx^{n-3}\ldots\ldots+Sx^{n-m+1}$.

146. Now P consists of the sum of all the quantities a, b, c, d, e, &c. the *number* of which is n; Q of the sum of the combinations of *any two* of those quantities, the number of which (by Art. 143) is $\frac{n(n-1)}{2}$; R of the sum of the combinations of *any three* of those quantities, the number of which is $\frac{n(n-1)(n-2)}{1.2.3}$; and S of the sum of the combinations of any $(m-1)$ of those quantities, the number of which is $\frac{n(n-1)(n-2)\ldots.(n-m+2)}{1.2.3\quad\ldots.\quad(m-1)}$.

147. Let $a=b=c=d=e$, &c. then $(x+a)(x+b)(x+c)(x+d)$, &c. becomes $(x+a)^n$, P becomes $n.a$, Q becomes $\frac{n(n-1)a^2}{2}$; R becomes $\frac{n(n-1)(n-2)}{1.2.3}a^3$, and S (the coefficient of the *m*th term becomes $\frac{n(n-1)(n-2)\ldots(n-m+2)}{1.2.3\ldots m-1}a^{m-1}$.

148. From hence it appears that $(x+a)^n=(x^n+Px^{n-1}+Qx^{n-2}+Rx^{n-3}\ .\ .\ .\ .\ .\ .\ +Sx^{n-m+1}=)\ x^n+nax^{n-1}+\frac{n(n-1)}{2}a^2x^{n-2}+\frac{n(n-1)(n-2)}{1.2.3}a^3x^{n-3}\ .\ .\ .\ .\ .\ .\ .\ .$

$$+\frac{n(n-1)(n-2)\ .\ .\ .\ .\ .\ (n-m+2)}{1.2.3\ .\ .\ .\ .\ .\ (m-1)}a^{m-1}x^{n-m+1};$$

and (preserving the notation adopted in Chap. III.) we have $(a+b)^n=a^n+na^{n-1}b+\frac{n(n-1)}{1.2}a^{n-2}b^2+$ &c. . . . . .

$$+\frac{n(n-1)(n-2)\ldots.(n-m+2)}{1.2.3\ldots.(m-1)}a^{n-m+1}b^{m-1}.$$

149. From the inspection of this series it is evident, that if, n be a *positive whole number*, it will *terminate* after $(n+1)$ terms; for let $m=n+2$, then $n-m+2=0$, and consequently the coefficient which involves the factor $(n-m+2)$ *vanishes*. Let $m=n+1$, then $n-m+2=1$, $n-m+1=0$, and $m-1=n$; $\therefore$ $(n+1\text{th})$ (or *last*) term is $\frac{n(n-1)(n-2)\ldots.3.2.1}{1.2.3\ \ldots.(n-1)n}$, a^0b^n or b^n. If n be *fractional* or *negative*, the series will not terminate, and in this case the value of any expanded binomial can only be expressed in the form of an *infinite series*.

150. If in the series expressing the value of $(a+b)^n$, for b we put $-b$, then those terms which involve the *odd* powers of b will be changed from $+$ to $-$; Hence we have,

$$(a+b)^n=a^n+na^{n-1}b+\frac{n(n-1)}{2}a^{n-2}b^2+\frac{n(n-1)(n-2)}{2.3}a^{n-3}b^3+\&c.$$

and $$(a-b)^n=a^n-na^{n-1}b+\frac{n(n-1)}{2}a^{n-2}b^2-\frac{n(n-1)(n-2)}{2.3}a^{n-3}b^3+\&c.$$

$\therefore$ by *addition*, $(a+b)^n+(a-b)^n=2a^n+n(n-1)a^{n-2}b^2+\&c.$

or $\frac{1}{2}(a+b)^n+\frac{1}{2}(a-b)^n=a^n+\frac{n(n-1)}{2}a^{n-2}b^2+\&c.$

by *subtraction*, $(a+b)^n-(a-b)^n=2na^{n-1}b+\frac{n(n-1)(n-2)}{3}a^{n-3}b^3+\&c.$

or $\frac{1}{2}(a+b)^n-\frac{1}{2}(a-b)^n=na^{n-1}b+\frac{n(n-1)(n-2)}{2.3}a^{n-3}b^3+\&c.$

151. Let $a=1$, $b=1$, then $(a+b)^n=(1+1)^n=2^n$; and since the several powers of a and b are, in this case, each of them equal to 1, we have $1+n+\frac{n(n-1)}{2}+\frac{n(n-1)(n-2)}{2.3}+$ &c. $=2^n$, i.e. *the sum of the coefficients of the* n*th power of any binomial is equal to that power of 2 whose index is* n. Thus, for the *square*, $1+2+1=4=2^2$; for the *cube*, $1+3+3+1=8=2^3$; for the *fourth* power, $1+4+6+4+1$

$+1=16=2^4$; &c. &c. If $a=1$, $b=1$, in the expression $(a-b)^n$, then $(1-1)^n=0$, which shews that the sum of the *positive* coefficients of $(a-b)^n$ is equal to the sum of the *negative* ones.

XLIII.

On the Expansion of Series.

152. It has already been observed (Art. 149) that when n is a *negative* number or a *fraction*, then the series expressing the value of $(a+b)^n$ does not terminate. Let $n=\frac{m}{r}$, and substitute $\frac{m}{r}$ for n in the series (Art. 148); then

$$\overline{a+b}|^{\frac{m}{r}}=a^{\frac{m}{r}}+\frac{m}{r}a^{\frac{m}{r}-1}b+\frac{\frac{m}{r}\left(\frac{m}{r}-1\right)}{2}a^{\frac{m}{r}-2}b^2+\frac{\frac{m}{r}\left(\frac{m}{r}-1\right)\left(\frac{m}{r}-2\right)}{2.3}a^{\frac{m}{r}-3}b^3+\&c.$$

$$=a^{\frac{m}{r}}+\frac{m\,a^{\frac{m}{r}}}{r}\left(\frac{b}{a}\right)+\frac{m(m-r)a^{\frac{m}{r}}}{2r^2}\left(\frac{b^2}{a^2}\right)+\frac{m(m-r)(m-2r)a^{\frac{m}{r}}}{2.3.r^3}\left(\frac{b^3}{a^3}\right)+\&c.^{(a)}$$

$$=a^{\frac{m}{r}}\left(1+\frac{m}{r}\left(\frac{b}{a}\right)+\frac{m(m-r)}{2r^2}\left(\frac{b^2}{a^2}\right)+\frac{m(m-r)(m-2r)}{2.3.r^3}\left(\frac{b}{a^3}\right)+\&c.\right)$$

which is a general expression for finding the *approximate* value of any binomial surd quantity, $\frac{m}{r}$ being either *positive* or negative, and m and r any whole numbers whatever.(a)

(a) This series is derived from the preceding one, by resolving the powers of a into two factors; thus $a^{\frac{m}{r}-1}=a^{\frac{m}{r}}\times a^{-1}=a^{\frac{m}{r}}\times\frac{1}{a}=\frac{a^{\frac{m}{r}}}{a}$; $a^{\frac{m}{r}-2}=a^{\frac{m}{r}}\times a^{-2}=a^{\frac{m}{r}}\times\frac{1}{a^2}=\frac{a^{\frac{m}{r}}}{a^2}$. See the Articles at the end of Chap. III.

EXAM. 1.

Find the approximate value of $\sqrt[3]{c^3+x^3}$ or $\overline{c^3+x^3}\big|^{\frac{1}{3}}$.

$$\left.\begin{array}{r}\text{Here } a=c^3\\ b=x^3\\ m=1\\ r=3\end{array}\right\}\quad\begin{array}{l}\therefore\ a^{\frac{m}{r}}=\sqrt[3]{c^3}=c;\\ \frac{m}{r}\left(\frac{b}{a}\right)=\frac{1}{3}\left(\frac{x^3}{c^3}\right)=\frac{x^3}{3c^3};\end{array}$$

$$\frac{m(m-r)}{2r^2}\left(\frac{b^2}{a^2}\right)=\frac{1(1-3)}{2.3^2}\left(\frac{x^6}{c^6}\right)=-\frac{x}{3^2.c^6};$$

$$\frac{m(m-r)(m-2r)}{2.3r^3}\left(\frac{b^3}{a^3}\right)=\frac{1(1-3)(1-6)}{2.3.3^3}\left(\frac{x^9}{c^9}\right)=\frac{5x^9}{3^4.c^9};$$

&c. = &c.

Hence $\sqrt[3]{c^3+x^3}=c\left(1+\frac{x^3}{3c^3}-\frac{x^6}{3^2.c^6}+\frac{5x^9}{3^4.c^9}-\text{\&c.}\right)$.

EXAM. 2.

Find the value of $\frac{d}{\sqrt{c^2+x^2}}$ or $\frac{d}{(c^2+x^2)^{\frac{1}{2}}}$ or $d(c^2+x^2)^{-\frac{1}{2}}$.

$$\left.\begin{array}{r}\text{Here } a=c^2\\ b=x^2\\ m=-1\\ r=2\end{array}\right\}\quad\begin{array}{l}\therefore\ a^{\frac{m}{r}}=c^{-1}=\frac{1}{c};\\ \frac{m}{r}\left(\frac{b}{a}\right)=-\frac{1}{2}\left(\frac{x^2}{c^2}\right)=-\frac{x^2}{2c^2};\end{array}$$

$$\frac{m(m-r)}{2r^2}\left(\frac{b^2}{a^2}\right)=\frac{-1(-1-2)}{2.2^2}\left(\frac{x^4}{c^4}\right)=\frac{3x^4}{2^3.c^4};$$

$$\frac{m(m-r)(m-2r)}{2.3r^3}\left(\frac{b^3}{a^3}\right)=\frac{-1(-1-2)(-1-4)}{2.3.2^3}\left(\frac{x}{c^6}\right)=-\frac{5x^6}{2^4.c^6};$$

&c. = &c.

Hence

(a) The proof of the Binomial Theorem, as given in Sect. XLII, goes upon the supposition that n is some positive whole number. To prove the truth of this Theorem when n is *negative* or *fractional*, requires a species of analytical investigation not well adapted to an Elementary Treatise. For a general demonstration of it, the Reader is therefore referred to BONNYCASTLE's *Algebra*, Vol. II. page 169.

$$\text{Hence } (c^2+x^2)^{-\frac{1}{2}}=\frac{1}{c}\left(1-\frac{x^2}{2c^2}+\frac{3x^4}{2^3.c^4}-\frac{5x^6}{2^4.c^6}+\&c.\right)$$

$$\text{and } \frac{d}{\sqrt{c^2+x^2}}=\frac{d}{c}\left(1-\frac{x^2}{2c^2}+\frac{3x^4}{2^3.c^4}-\frac{5x^6}{2^4.c^6}+\&c.\right)$$

EXAM. 3.

Find the value of $\frac{1}{(c+x)^2}$ or $(c+x)^{-2}$ in a series.

$$\text{Here } \left.\begin{array}{r} a=c \\ b=x \\ m=-2 \\ r=1 \end{array}\right\} \begin{array}{l} \therefore\ a^{\frac{m}{r}}=c^{-2}=\frac{1}{c^2}; \\ \frac{m}{r}\left(\frac{b}{a}\right)=-\frac{2x}{c}; \end{array}$$

$$\frac{m(m-r)}{2r^2}\left(\frac{b^2}{a^2}\right)=\frac{-2(-2-1)}{2}\left(\frac{x^2}{c^2}\right)=\frac{3x^2}{c^2};$$

$$\frac{m(m-r)(m-2r)}{2.3r^3}\left(\frac{b^3}{a^3}\right)=\frac{-2(-2-1)(-2-2)}{2.3}\left(\frac{x^3}{c^3}\right)=-\frac{4x^3}{c^3};$$

$$\&c.=\&c.$$

$$\text{Hence } \frac{1}{(c+x)^2}=\frac{1}{c^2}\left(1-\frac{2x}{c}+\frac{3x^2}{c^2}-\frac{4x^3}{c^3}+\&c.\right)$$

EXAM. 4.

Find the value of $(c^2-x^2)^{\frac{3}{4}}$ in a series.

$$\text{Here } \left.\begin{array}{r} a=c^2 \\ b=-x^2 \\ m=3 \\ r=4 \end{array}\right\} \begin{array}{l} \therefore\ a^{\frac{m}{r}}=\sqrt[4]{c^6}=\sqrt{c^3}; \\ \frac{m}{r}\left(\frac{b}{a}\right)=\frac{3}{4}\left(-\frac{x^2}{c^2}\right)=-\frac{3x^2}{2^2.c^2}; \end{array}$$

$$\frac{m(m-r)}{2r^2}\left(\frac{b^2}{a^2}\right)=\frac{3(3-4)}{2.4^2}\left(\frac{x^4}{c^4}\right)=-\frac{3x^4}{2^5.c^4}$$

$$\frac{m(m-r)(m-2r)}{2.3r^3}\left(\frac{b^3}{a^3}\right)=\frac{3(3-4)(3-8)}{2.3.4^3}\left(\frac{x^6}{c^6}\right)=-\frac{5x^6}{2^7.c^6};$$

$$\&c.=\&c.$$

$$\text{Hence } (c^2-x^2)^{\frac{3}{4}}=\sqrt{c^3}\left(1-\frac{3x^2}{2^2.c^2}-\frac{3x^4}{2^5.c^4}-\frac{5x^6}{2^7.c^6}-\&c.\right)$$

153. Now let $m=1$, then $(a+b)^{\frac{m}{r}}=(a+b)^{\frac{1}{r}}=\sqrt[r]{a+b}$; and $a^{\frac{m}{r}}=\sqrt[r]{a}$; hence the series in Art. 152 is transformed into

$$\sqrt[r]{a+b}=\sqrt[r]{a}\left(1+\frac{1}{r}\left(\frac{b}{a}\right)+\frac{1-r}{2r^2}\left(\frac{b^2}{a^2}\right)+\frac{(1-r)(1-2r)}{2.3.r^3}\left(\frac{b^3}{a^3}\right)+\frac{(1-r)(1-2r)(1-3r)}{2.3.4r^4}\left(\frac{b^4}{a^4}\right)+\&c.\right)(A).$$

Let $a=1$, $b=1$, then

$$\sqrt[r]{2}=1+\frac{1}{r}+\frac{1-r}{2r^2}+\frac{(1-r)(1-2r)}{2.3r^3}+\frac{(1-r)(1-2r)(1-3r)}{2.3.4.r^4}+\&c.\ (B).$$ Thus

If $r=2$, then $\sqrt{2}=1+\frac{1}{2}-\frac{1}{2^3}+\frac{1}{2^4}-\frac{5}{2^7}+\frac{7}{2^8}-\frac{3.7}{2^{10}}+\frac{3.11}{2^{12}}-\&c.$

$r=3, \ldots \sqrt[3]{2}=1+\frac{1}{3}-\frac{1}{3^2}+\frac{5}{3^4}-\frac{2.5}{3^5}+\frac{2.11}{3^6}-\frac{2.7.11}{3^8}+\&c.$

$\&c.=\&c.$

By means of the series marked (A), the rth root of many other numbers may be found, if a and b be so assumed, that b is a small number with respect to a, and $\sqrt[r]{a}$ a whole number; thus,

EXAMPLE 1.

Let $a=4$, $b=1$, $r=2$, then $\sqrt[r]{a}=\sqrt{4}=2$, and we have

$$\sqrt{4+1}=\sqrt{5}=2\left(1+\frac{1}{2^3}-\frac{1}{2^7}+\frac{1}{2^{10}}-\frac{5}{2^{15}}+\&c.\right).$$

EXAM. 2.

Let $a=8$, $b=1$, $r=3$, then $\sqrt[r]{a}=\sqrt[3]{8}=2$, and we obtain

$$\sqrt[3]{8+1}=\sqrt[3]{9}=2\left(1+\frac{1}{3.8}-\frac{1}{3^2.8^2}+\frac{5}{3^4.8^3}-\frac{2.5}{3^5.8^4}+\&c.\right)$$

EXAM. 3.

EXAM. 3.

Let $a=8$, $b=-2$, $r=3$, then $\frac{b}{a}=-\frac{2}{8}=-\frac{1}{4}$, and we have

$$\sqrt{8-2}=\sqrt[3]{6}=2\left(1-\frac{1}{3.4}-\frac{1}{3^2.4^2}-\frac{5}{3^4.4^3}-\frac{2.5}{3^5.4^4}+\&c.\right).$$

The several terms of these series are found by substituting for a, b, and r their values in the general series marked (A) or (B), and then rejecting the factors common to both the numerators and denominators of the fractions. Thus, for instance, to find the *seventh* term of the series exhibiting the value of $\sqrt{2}$, we take the 7th term of the series marked (B), which is $\frac{(1-r)(1-2r)(1-3r)(1-4r)(1-5r)}{2.3.4.5.6r^6}$; and since $r=2$, the fraction is $-\frac{3.5.7.9}{2.3.4.5.6.2^6}=-\frac{7.9}{2.4.6.2^6}=\left(\text{for } \frac{9}{6}=\frac{3}{2}\right)$ $-\frac{3.7}{2.4.2.2^6}=-\frac{3.7}{2^{10}}$. To find the 5th term of the series expressing the approximate value of $\sqrt[3]{9}$, we take the 5th term of the general series marked (A), which is $\frac{(1-r)(1-2r)(1-3r)}{2.3.4r^4}$ $\left(\frac{b^4}{a^4}\right)$, where $a=8$, $b=1$, and $r=3$; $\therefore$ the value of the fraction is $-\frac{2.5.8}{2.3.4.3^4}\left(\frac{1}{8^4}\right)=-\frac{2.5}{2.3.4.3^4.8^3}=-\frac{2.5}{3.3^4.8.8^3}=$ $-\frac{2.5}{3^5.8^4}$. In this manner each term of the several series is calculated; and the law which they observe is, that the *numerators* of the fractions consist of certain combinations of prime numbers, and the *denominators* of combinations of certain powers of a and r.

154. These series converge very fast, so that a few terms would give the rth root of certain numbers with a great degree of accuracy. But a more practical method of finding the higher roots of such numbers, is, by making the number whose root is

to be extracted equal to a^r+b, and then assuming $a+x=\sqrt[r]{a^r+b}$, x being some decimal fraction; for in this case $(a+x)^r=a^r+b$, and by expanding $(a+x)^r$ and neglecting all the powers of x after x^2 (being very small compared with the preceding ones) we have

$$a^r+ra^{r-1}x+r\left(\frac{r-1}{2}\right)a^{r-2}x^2=a^r+b;$$

$$\therefore\ ra^{r-1}x+r\left(\frac{r-1}{2}\right)a^{r-2}x^2=b\ (A)$$ an equation from which the value of x may be found in two ways.

I. By arranging the terms, and dividing by $r\left(\frac{r-1}{2}\right)a^{r-2}$, we have

$$x^2+\frac{2ax}{r-1}=\frac{2b}{r(r-1)a^{r-2}};$$

$$\therefore\ x^2+\frac{2ax}{r-1}+\frac{a^2}{(r-1)^2}=\frac{2b}{r(r-1)a^{r-2}}+\frac{a^2}{(r-1)^2};$$

and by solving the quadratic,

$$x=-\frac{a}{r-1}+\sqrt{\frac{2b}{r(r-1)a^{r-2}}+\frac{a^2}{(r-1)^2}}.$$

Hence $\sqrt[r]{a^r+b}=a+x=\frac{r-2}{r-1}a+\sqrt{\frac{2b}{r(r-1)a^{r-2}}+\frac{a^2}{(r-1)^2}}$, which is HALLEY's Rule, (*Philosophical Transactions*, 1694).

II. From equation (*A*) we have $x\left(ra^{r-1}+r\left(\frac{r-1}{2}\right)a^{r-2}x\right)=b$,

$$\therefore\ x=\frac{b}{ra^{r-1}+r\left(\frac{r-1}{2}\right)a^{r-2}x}=\frac{b}{ra^{r-2}}\left(\frac{1}{a+\frac{r-1}{2}x}\right).$$

By a *first* approximation, neglecting the term which involves x, we have $x=\frac{b}{ra^{r-1}}$; substitute this value for x in the fraction $\frac{b}{ra^{r-2}}\left(\frac{1}{a+\frac{r-1}{2}.x}\right)$, and we obtain a *second* approximation which gives $$x=\frac{b}{ra^{r-2}}\left(\frac{1}{a+\frac{r-2}{2r}\left(\frac{b}{a^{r-1}}\right)}\right);$$

and $\sqrt[r]{a^r+b}=a+x=a+\frac{b}{ra^{r-2}}\left(\frac{1}{a+\frac{r-2}{2r}\left(\frac{b}{a^{r-1}}\right)}\right)$; which is the Rule given by LA CROIX (*Complémens d'Algébre*), and ascribed to LAMBERT.

EXAM. 1.

Find an approximate value of the cube root of 67.

Now $67=64+3=4^3+3$; $\therefore a=4,\ b=3,\ r=3$; hence, by the *first* method, $a+x=\frac{1}{2}a+\sqrt{\frac{b}{3a}+\frac{a^2}{4}}$; or $\sqrt[3]{67}=2+\sqrt{\frac{1}{4}+4}=2+2.0615=4.0615.$

EXAM. 2.

Find an approximate value of the fifth root of 30.

Here $30=32-2=2^5-2$; $\therefore a=2,\ b=-2,\ r=5$; hence, by the *second* method, $a+x=a-\frac{b}{5a^3}\left(\frac{1}{a-\frac{3b}{10a^4}}\right)$, or

$$\sqrt[5]{30}=2-\frac{1}{20}\left(\frac{1}{2-\frac{3}{80}}\right)=2-\frac{4}{157}=\frac{310}{157}=1.9745.$$

The method of finding the rth root of certain numbers as exhibited in this and the foregoing Article, is a matter rather of curiosity than practical utility, as the rth root of any number whatever may be found with great facility by means of *Logarithms.* This method would be useful, however, in an operation where it was required to express this root in the form of a *vulgar fraction;* as in the last Example, where we obtained the approximate value of the 5th root of 30 in the shape of the fraction $\frac{310}{157}$.

XLIV.

XLIV.

On the method of approximating to the Powers and Roots of Numbers whose Difference is small.

155. Let $a+x$ and a be two numbers whose difference is x, then $(a+x)^n : a^n :: a^n + na^{n-1}x + \frac{n(n-1)}{2}a^{n-2}x^2 + \frac{n(n-1)(n-2)}{2.3} a^{n-3}x^3 + \&c. : a^n ::$ (dividing each term of the ratio by a^{n-1}) $a + nx + \frac{n(n-1)}{2}\left(\frac{x^2}{a}\right) + \frac{n(n-1)(n-2)}{2.3}\left(\frac{x^3}{a^2}\right) + \&c. : a$.

156. Suppose now that n is not a large number, and that x is very small when compared with a, then the fractions $\frac{x^2}{a}, \frac{x^3}{a^2}$, &c. will be small also, and those terms in which they are involved will be very small when compared with the *integral* part $a+nx$ of the series; in this case, therefore, the ratio of $(a+x)^n : a^n$ approximates to the ratio of $a+nx : a$. Thus the ratio of $(a+x)^2 : a^2$ approximates to the ratio of $a+2x : a$; of $(a+x)^3 : a^3$ to the ratio of $a+3x : a$; &c. &c.; or if $n=\frac{1}{2}, \frac{1}{3}$, &c. then the ratio of $\sqrt{a+x} : a$ approximates to the ratio of $a+\frac{1}{2}x : a$; of $\sqrt[3]{a+x} : a$ to the ratio of $a+\frac{1}{3}x : a$; &c. &c. For instance, the ratio of the *square* of 501 to the square of 500 (in which case, $a=500$, $x=1$, $n=2$) is 502 : 500 very nearly; the ratio of the *cube* of 62 to the cube of 61, is 64 : 61 very nearly; &c. &c. Again, the ratio of the *square root* of 501 to the square root of 500 is $500\frac{1}{2}$: 500; and of the *cube root* of 103 to the cube root of 100, is 101 : 100, very nearly.

157. If the difference between the two numbers is not very small when compared with the numbers themselves, then the *three* first terms of the series must be taken instead of *two*, in which case the approximate ratio of $(a+x)^n : a^n$ becomes that of $a+nx+\frac{n(n-1)}{2}\left(\frac{x^2}{a}\right) : a$. For instance, let it be required to

find a near approximation to the ratio of $\sqrt[3]{11} : \sqrt[3]{10}$, then $a=10$, $x=1$, $n=\frac{1}{3}$, and the approximate ratio becomes that of $10+\frac{1}{3}-\frac{1}{90} : 10$, or of $\frac{900+30-1}{90} : 10$, or of $929 : 900$. By the Theorem in Art. 156, this approximation would be $10\frac{1}{3} : 10$, or $31 : 30$, i.e. $930 : 900$.

XLV.

On the method of extracting the nth *Root of a Binomial Quadratic Surd.*

158. Let $x+\sqrt{y}$ be any binomial quadratic surd, in which x is a rational quantity and $\sqrt{y}$ an irreducible surd, then $(x+\sqrt{y})^n = x^n + nx^{n-1}\sqrt{y} + n.\frac{n-1}{2}x^{n-2}y + n.\frac{n-1}{2}.\frac{n-2}{3}x^{n-3}y\sqrt{y} + \&c.\ (P)$. Let the sum of the *rational* terms in the series (P) be equal to a, and of the *irrational* $=p\sqrt{y}=$ (Art. 121.) $\sqrt{p^2y}$, which may be expressed in the form $\sqrt{b}$, $\sqrt{b}$ being a quadratic surd containing the irreducible surd $\sqrt{y}$. Hence $(x+\sqrt{y})^n=a+\sqrt{b}$, and $\sqrt[n]{a+\sqrt{b}}=x+\sqrt{y}$; if, therefore, the *n*th root of a quadratic surd of the form $a+\sqrt{b}$ can be extracted, it may be expressed under the form $x+\sqrt{y}$, whether n be an *odd* or *even* number.

159. Let $\sqrt{x}+\sqrt{y}$ be a binomial quadratic surd, in which $\sqrt{x}$ and $\sqrt{y}$ are irreducible surds, then $(\sqrt{x}+\sqrt{y})^n= x^{\frac{n}{2}} + nx^{\frac{n-1}{2}}\sqrt{y} + n.\frac{n-1}{2}x^{\frac{n-2}{2}}y + n.\frac{n-1}{2}.\frac{n-2}{3}x^{\frac{n-3}{2}}y\sqrt{y} + \&c.\ (Q)$. If n be an *even* number, then the 1st, 3d, 5th, &c. terms of the series (Q) are *rational*, and the 2d, 4th, 6th, &c. *irrational*, and $(\sqrt{x}+\sqrt{y})^n$ may (as before) be expressed under the form $a+\sqrt{b}$, where $\sqrt{b}$ is a quadratic surd containing the irreducible surd $\sqrt{y}$. Hence $(\sqrt{x}+\sqrt{y})^n=a+\sqrt{b}$, or $\sqrt[n]{a+\sqrt{b}}$ $=\sqrt{x}$

$=\sqrt{x}+\sqrt{y}$; and, if the *n*th root of $a+\sqrt{b}$ can be extracted, it may be expressed under the form $\sqrt{x}+\sqrt{y}$.

160. If n be an *odd* number, then $\frac{n}{2}, \frac{n-2}{2}$, &c. are *fractions*, and $\frac{n-1}{2}, \frac{n-3}{2}$, &c. *whole numbers*; hence the 1st, 3d, 5th, &c. terms of the series (Q) will be surd quantities involving $\sqrt{x}$, and the 2d, 4th, 6th, &c. terms, surd quantities involving $\sqrt{y}$; the series may therefore be expressed under the form $p\sqrt{x}+q\sqrt{y} =\sqrt{p^2x}+\sqrt{q^2y}$, or under the more general form $\sqrt{a}+\sqrt{b}$, where $\sqrt{a}$ and $\sqrt{b}$ are quadratic surds involving the irreducible surds $\sqrt{x}$ and $\sqrt{y}$ respectively. In this case, then, $(\sqrt{x}+\sqrt{y})^n =\sqrt{a}+\sqrt{b}$, or $\sqrt[n]{\sqrt{a}+\sqrt{b}}=\sqrt{x}+\sqrt{y}$; and consequently, if the *n*th root of $\sqrt{a}+\sqrt{b}$ can be extracted, it may be expressed under the form $\sqrt{x}+\sqrt{y}$.

161. From hence it appears that the *n*th root of $a+\ b$ may be expressed under the form $x+\sqrt{y}$, whether n be an *odd* or an *even* number; that the *n*th root of $a+\sqrt{b}$ may *also* be expressed under the form $\sqrt{x}+\sqrt{y}$, when n is an *even* number; but that the *n*th root of $\sqrt{a}+\sqrt{b}$ can be expressed in the form of a binomial quadratic surd only when n is an *odd* number, and then under the form $\sqrt{x}+\sqrt{y}$.

162. Suppose now that $\sqrt[n]{a+\sqrt{b}}=x+\sqrt{y}$, then $a+\sqrt{b}= (x+\sqrt{y})^n=x^n+nx^{n-1}\sqrt{y}+n.\frac{n-1}{2}x^{n-2}y+n.\frac{n-1}{2}.\frac{n-2}{3}x^{n-3} y\sqrt{y}+$ &c.; $\therefore$ by Art. 135, $a=x^n+n.\frac{n-2}{2}x^{n-2}y+$ &c. and $\sqrt{b}=nx^{n-1}\sqrt{y}+n.\frac{n-1}{2}.\frac{n-2}{3}x^{n-3}y\sqrt{y}+$ &c.; hence $a-\sqrt{b} =x^n-nx^{n-1}\sqrt{y}+n.\frac{n-1}{2}x^{n-3}y-n.\frac{n-1}{2}.\frac{n-2}{3}x^{n-2}y\sqrt{y}+$ &c. $=(x-\sqrt{y})^n$, or $\sqrt[n]{a-\sqrt{b}}=x-\sqrt{y}$; from which it appears that if $\sqrt[n]{a+\sqrt{b}}=x+\sqrt{y}$, then will $\sqrt[n]{a-\sqrt{b}}=x-\sqrt{y}$.

163. Let $\sqrt[n]{\sqrt{a}+\sqrt{b}}=\sqrt{x}+\sqrt{y}$ (n being an *odd* number), then $\sqrt{a}+\sqrt{b}=(\sqrt{x}+\sqrt{y})^n=x^{\frac{n}{2}}+nx^{\frac{n-1}{2}}\sqrt{y}+n.\frac{n-1}{2}x^{\frac{n-2}{2}}y+n.\frac{n-1}{2}.\frac{n-2}{3}x^{\frac{n-3}{2}}y\sqrt{y}+$ &c.; hence by Art. 135. (since $\sqrt{a}$ is a quadratic surd involving $\sqrt{x}$, and $\sqrt{b}$ a quadratic surd involving $\sqrt{y}$) $\sqrt{a}=x^{\frac{n}{2}}+n.\frac{n-1}{2}x^{\frac{n-2}{2}}y+$&c. and $\sqrt{b}=nx^{\frac{n-1}{2}}\sqrt{y}+n.\frac{n-1}{2}.\frac{n-2}{3}x^{\frac{n-3}{2}}y\sqrt{y}+$&c.; $\therefore\ \sqrt{a}-\sqrt{b}=x^{\frac{n}{2}}-nx^{\frac{n-1}{2}}\sqrt{y}+n.\frac{n-1}{2}x^{\frac{n-2}{2}}y-n.\frac{n-1}{2}.\frac{n-2}{3}x^{\frac{n-3}{2}}y\sqrt{y}+$&c. $=(\sqrt{x}-\sqrt{y})^n$, or $\sqrt[n]{\sqrt{a}-\sqrt{b}}=\sqrt{x}-\sqrt{y}$; from which it follows, that if $\sqrt[n]{\sqrt{a}+\sqrt{b}}=\sqrt{x}+\sqrt{y}$, then $\sqrt[n]{\sqrt{a}-\sqrt{b}}=\sqrt{x}-\sqrt{y}$.

164. Suppose now that $A+B$ is a binomial quadratic surd, one or both of whose terms is irrational, then, from what has been shewn, it appears that if A and B are both irrational, the nth root of $A+B$ can be extracted only when n is an *odd* number; but if A be *rational*, then the nth root of $A+B$ may be extracted, whether n be an *odd* or an *even* number. In the following Rule for extracting the nth root of $A+B$, the two terms are supposed to be so arranged that A is greater than B; and it consists of two cases, depending upon the value of A^2-B^2.

CASE I.

When A^2-B^2 is a complete nth power; i.e. when A^2-B $=\alpha^n$ or $\sqrt[n]{A^2-B^2}=\alpha$, α being some whole number.

$$\text{Assume } \sqrt[n]{A+B}=\sqrt{x}+\sqrt{y}\ (R),$$

$$\text{then (by Art. 163) } \sqrt[n]{A-B}=\sqrt{x}-\sqrt{y}\ (S);$$

$$\therefore\ \sqrt[n]{A^2-B^2}\ (\alpha)=x-y.$$

$$\begin{array}{ll}\text{By squaring equation } (R) & \sqrt{A^2+B^2+2AB}=x+y+2\sqrt{xy}, \\ \ldots\ldots\ \text{equation } (S) & \sqrt[n]{A^2+B^2-2AB}=x+y-2\sqrt{xy}; \\ \hline \end{array}$$

$\therefore \sqrt[n]{A^2+B^2+2AB}+\sqrt{A^2+B^2-2AB}=2x+2y=$ some *whole number.*

Now let $\sqrt{A^2+B^2+2AB}$ in the nearest whole number $=p+f$ (f being a proper fraction);

and let $\sqrt[n]{A^2+B^2-2AB}$ (q being a whole number, and f' a proper fraction) $=q+f'$.

Then $\sqrt[n]{A^2+B^2+2AB}+\sqrt[n]{A^2+B^2-2AB}=p+q+f+f'=2x+2y=$ a whole number, $\therefore f+f'=0$,[a] or $f'=-f$; hence, if p be the nearest whole number *less* than $\sqrt[n]{A^2+B^2+2AB}$, and q the nearest whole number *greater* than $\sqrt[n]{A^2+B^2-2AB}$, the integral value of $\sqrt[n]{A^2+B^2+2AB}+\sqrt[n]{A^2+B^2-2AB}$ will be $p+q$; let $p+q=t$, then

$$\left.\begin{array}{r}2x+2y=t \text{ or } x+y=\frac{1}{2}t \\ \text{but } x-y=\alpha\end{array}\right\} \begin{array}{l}\therefore 2x=\frac{1}{2}t+\alpha, \text{and } \sqrt{x}=\frac{1}{2}\sqrt{t+2\alpha}, \\ 2y=\frac{1}{2}t-\alpha, \text{and } \sqrt{y}=\frac{1}{2}\sqrt{t-2\alpha}.\end{array}$$

Hence $\sqrt[n]{A\pm B}=\sqrt{x}\pm\sqrt{y}=\frac{1}{2}\sqrt{t+2\alpha}\pm\frac{1}{2}\sqrt{t-2\alpha}$.

CASE II.

When A^2-B^2 is not a complete *n*th power.

In this case let C be so assumed as to make A^2-B^2 a complete *n*th power, i.e. let $(A^2-B^2)C=\alpha^n$, or $\sqrt[n]{(A^2-B^2)C}=\alpha$; then assume

---

(a) Since p and q are both positive whole numbers, and f, f' are proper fractions, there is no other supposition which can make $p+q+f+f'$ a *whole number*, but that of $f+f'$ being equal to 0, or $f'=-f$; hence if $p+f$ be the real value of $\sqrt[n]{A^2+B^2+2AB}$, $q-f$ will be the real value of $\sqrt[n]{A^2+B^2-2AB}$.

assume $\sqrt[n]{(A+B)\sqrt{C}}=\sqrt{x}+\sqrt{y}$, or $\sqrt[n]{A+B}=\frac{\sqrt{x}+\sqrt{y}}{\sqrt[2n]{C}}$;

$$\therefore\ \sqrt[n]{(A-B)\sqrt{C}}=\sqrt{x}-\sqrt{y};$$

or $\sqrt[n]{(A^2-B^2)C}\,(\alpha)=x-y$; and,

$$\sqrt[n]{(A^2+B^2+2AB)C}+\sqrt[n]{(A^2+B^2-2AB)C}=2x+2y=t.$$

From which we deduce, as before,

$$\sqrt{x}\pm\sqrt{y}=\tfrac{1}{2}\sqrt{t+2\alpha}\pm\tfrac{1}{2}\sqrt{t-2a};$$

$$\therefore\ \sqrt[n]{A\pm B}=\frac{\sqrt{x}+\sqrt{y}}{\sqrt[2n]{C}}=\frac{\sqrt{t+2\alpha}\pm\sqrt{t-2\alpha}}{2\sqrt[2n]{C}}.$$

EXAM. 1.

Find the cube root of $26+15\sqrt{3}$.

Here $\left.\begin{array}{l}A=26\\ B=15\sqrt{3}\end{array}\right\}\therefore A^2-B^2=676-675=1$, and $\alpha=1$.

$$\sqrt[3]{A^2+B^2+2AB}=\sqrt[3]{676+675+780\sqrt{3}}=13+f,$$

$$\sqrt[3]{A^2+B^2-2AB}=\sqrt[3]{676+675-780\sqrt{3}}=1-f;$$

$$\therefore t=13+1=14.$$

Hence $\sqrt[3]{A+B}=\frac{1}{2}\sqrt{t+2\alpha}+\frac{1}{2}\sqrt{t-2\alpha}=\frac{1}{2}\sqrt{16}+\frac{1}{2}\sqrt{12}$ $=2+\sqrt{3}$.

EXAM. 2.

Find the cube root of $9\sqrt{3}-11\sqrt{2}$.

Here $\left.\begin{array}{l}A=9\sqrt{3}\\ B=11\sqrt{2}\end{array}\right\}\therefore A^2-B^2=243-242=1$, and $\alpha=1$,

$$\sqrt[3]{A^2+B^2+2AB}=\sqrt[3]{243+242+198\sqrt{6}}=9+f,$$

$$\sqrt[3]{A^2+B^2-2AB}=\sqrt[3]{243+242-198\sqrt{6}}=1-f.$$

Hence $t=9+1=10$, and $\frac{1}{2}\sqrt{t+2\alpha}-\frac{1}{2}\sqrt{t-2\alpha}=\frac{1}{2}\sqrt{12}-\frac{1}{2}\sqrt{8}=\sqrt{3}-\sqrt{2}$.

$$\therefore\ \sqrt[3]{A-B}=\sqrt{3}-\sqrt{2}.$$

EXAM. 3.

EXAM. 3.

Find the cube root of $8+4\sqrt{5}$ or $4\sqrt{5}+8$.

Here $\left.\begin{array}{l}A=4\sqrt{5}\\ B=8\end{array}\right\}\therefore A^2-B^2=80-64=16$, which is not a *cube* number, and the least number which multiplied into it will produce a cube number is 4,(a) $\therefore C=4$, and $(A^2-B^2)C=16\times4=64$; hence $\alpha^3=64$, and $\alpha=4$.

Now $\sqrt[3]{(A^2+B^2+2AB)C}=\sqrt[3]{(80+64+64\sqrt{5})4}=10+f$,

$\sqrt[3]{(A^2+B^2-2AB)C}=\sqrt[3]{(80+64-64\sqrt{5})4}=2-f$;

$$\therefore t=10+2=12,$$

and $\dfrac{\sqrt{t+2\alpha}+\sqrt{t-2\alpha}}{2\sqrt[2n]{C}}=\dfrac{\sqrt{20}+\sqrt{4}}{2\sqrt[6]{4}}=\dfrac{\sqrt{5}+1}{\sqrt[3]{2}}$.

Hence $\sqrt[3]{4\sqrt{5}+8}=\dfrac{\sqrt{5}+1}{\sqrt[3]{2}}$.

(a) In finding the *least* number by which a given number (a) must be multiplied so as to give a product which shall be a complete *n*th power, it may be observed, that if a be a *prime* number, it must always be multiplied by a^{n-1}; thus, there is no other number by which 3 can be multiplied to make it a *cube* number, but 3^2 or 9, which gives the product 27; nor is there any other number by which 5 can be multiplied to make it a *biquadrate* number, but 5^3 or 125, which gives the product 625. But if the given number is resolvable into factors, one or more of which are *square*, *cube*, &c. numbers, then a *less* number than a^{n-1} will answer the purpose. Thus $12=3\times4=3\times2^2$; and if 3×2 be multiplied by $3^2\times2$, it gives $3^3\times2^3$, which is the cube of 3×2; i.e. if 12 be multiplied by 18 it gives 216 the cube of 6. Or in general, if the given number (a) be resolvable into factors α, β, γ, &c. such that $a=\alpha^m\beta^p\gamma^q$ &c. then if this number be multiplied by $\alpha^{n-m}\beta^{n-p}\gamma^{n-q}$ &c. it gives $\alpha^n\beta^n\gamma^n$ &c. which is the *n*th power of $\alpha\beta\gamma$ &c. Thus $360=8\times9\times5=2^3\times3^2\times5$; here $m=3$, $p=2$, $q=1$; and if it be required to find a multiplier which should make it a *biquadrate* number, then $n=4$, $\therefore n-m=1$, $n-p=2$, $n-q=3$; hence the multiplier is $2\times3^2\times5^3=2250$, and we have $360\times2250=810000$, which is the fourth power of $2\times3\times5$ or 30. If one or more of the indices m, p, q, &c. be *greater* than n, then, in finding the multiplier, such multiple of n must be taken as to make the indices of all the factors in the multiplier *positive*; thus if m be greater than n but less than $2n$, then the multiplier to be taken is $\alpha^{2n-m}\beta^{n-p}\gamma^{n-q}$, which gives for the product of it and $\alpha^m\beta^p\gamma^q$ the quantity $\alpha^{2n}\beta^n\gamma^n$, which is the *n*th power of $\alpha^2\beta\gamma$.

XLVI.

On the Comparison and Reversion of Series.

Our observations upon the "comparison and reversion of series," will be confined to such as are of the form $ax+bx^2+cx^3+dx^4+ex^5+$&c. referring the Reader to the 10th Chap. of the 2d Part of MACLAURIN's *Algebra*, and to other authors who have written upon the more advanced parts of the science, for the further illustration of the subject.

165. Previous to the reversion of this series, it will be necessary to shew the manner in which it may be raised to any power (n). This is done by separating the first term from the rest, and then applying the binomial theorem to the involution of the series so transformed; thus

$$(ax+bx^2+cx^3+dx^4+\&c.)^n=\overline{ax+(bx^2+cx^3+dx^4+\&c.)}\Big|^n$$

$$=a^nx^n+na^{n-1}x^{n-1}(bx^2+cx^3+dx^4+\&c.)+\frac{n(n-1)}{2}a^{n-2}x^{n-2}$$
$$(bx^2+cx^3+\&c.)^2+\&c.$$

$$=a^nx^n+na^{n-1}x^{n-1}(bx^2+cx^3+dx^4+\&c.)+\frac{n(n-1)}{2}a^{n-2}x^{n-2}$$
$$(b^2x^4+2bcx^5+\&c.)+\&c.$$

$$=a^nx^n+na^{n-1}bx^{n+1}+\left.\begin{matrix}na^{n-1}c\\+\frac{n(n-1)}{2}a^{n-2}b^2\end{matrix}\right\}x^{n+2}\left.\begin{matrix}+na^{n-1}d\\+n(n-1)a^{n-2}bc\end{matrix}\right\}x^{n+3}+\&c.$$

We have here found only the four first terms of the involved series, but they are sufficient for our present purpose.

166. Let $ax+bx^2+cx^3+dx^4+$&c. and $\alpha x+\beta x^2+\gamma x^3+$ x^4+&c. be two series of this kind, in which a, b, c, d, &c. α, β, γ, δ, &c. are invariable coefficients; if these two series be equal to each other, then will $a=\alpha$; $b=\beta$; $c=\gamma$; $d=\delta$; &c. For, let the terms of each series be divided by x, then $a+bx+cx^2+dx^3+$&c.$=\alpha+\beta x+\gamma x^2+\delta x^3+$&c.; and since x may vary through all degrees of magnitude whilst the coefficients

remain constant, let $x=0$, then $a=\alpha$; but a and α are invariable quantities; if therefore they are once equal, they will always be equal. Again, since $a=\alpha$, we have $bx+cx^2+dx^3$ + &c. $=\beta x+\gamma x^2+\delta x^3+$ &c.; divide each side of the equation by x, then $b+cx+dx^2+$ &c. $=\beta+\gamma x+\delta x^2+$ &c.; let $x=0$, then $b=\beta$; and so we might proceed till all the terms of the series were exhausted; we therefore infer, that when these two series are equal to each other, $a=\alpha$; $b=\beta$; $c=\gamma$; $d=\delta$; &c. &c.

167. By transposing the terms of the equation $ax+bx^2+cx^3+dx^4+$ &c. $=\alpha x+\beta x^2+\gamma x^3+\delta x^4+$ &c. we have $(a-\alpha)x+(b-\beta)x^2+(c-\gamma)x^3+(d-\delta)x^4+$ &c. $=0$. Now whatever is true in the original equation, must also be true in the transposed equation; but it has already been proved, with respect to the former equation, that $a=\alpha$; $b=\beta$; $c=\gamma$; $d=\delta$; &c.; hence $a-\alpha=0$; $b-\beta=0$; $c-\gamma=0$; $d-\delta=0$; &c.; from which it follows that in an equation of the form $(a-\alpha)x+(b-\beta)x^2+(c-\gamma)x^3+(d-\delta)x^4+$ &c. $=0$, whose coefficients consist of *positive* and *negative* quantities, these coefficients must all become equal to 0 at the same time.

168. Suppose now that $x=ay+by^2+cy^3+dy^4+$ &c. and that it is required to find the value of y in the terms of x. Transpose x to the other side of the equation, then $ay+by^2+cy^3+dy^4+$ &c. $-x=0$. Assume $y=\alpha x+\beta x^2+\gamma x^3+\delta x^4+$ &c.; and finding the value of the successive powers of y, by Art. 165, we have

$$\left.\begin{array}{ll}
ay = & a\alpha x+a\beta x^2+a\gamma x^3+a\delta x^4+\text{\&c.} \\
by^2 = & \left.\begin{array}{l} b\alpha^2x^2+2b\alpha\beta x^3+2b\alpha\gamma x^4+\text{\&c.} \\ \qquad\qquad\qquad +b\beta^2x^4+\text{\&c.}\end{array}\right\} \\
cy^3 = & \ldots\ldots\ c\alpha^3x^3+3c\alpha^2\beta x^4+\text{\&c.} \\
dy^4 = & \ldots\ldots\ldots\ d\alpha^4x^4+\text{\&c.} \\
\text{\&c.} = & \ldots\ldots\ldots\ldots\ +\text{\&c.} \\
-x = & -x
\end{array}\right\}=0.$$

Hence, by Art. 167, $a\alpha-1=0$, or $\alpha=\frac{1}{a}$;

$$a\beta+b\alpha^2=0, \text{ or } \beta=-\frac{b\alpha^2}{a}=-\frac{b}{a^3};$$

$$a\gamma+2b\alpha\beta+c\alpha^3=0, \text{ or } \gamma=\frac{-2b\alpha\beta-c\alpha^3}{a}=\frac{2b^2-ac}{a^5};$$

$$a\delta+2b\alpha\gamma+b\beta^2+3c\alpha^2\beta+d\alpha^4=0, \text{ or } \delta=\frac{-2b\alpha\gamma-b\beta^2-3c\alpha^2\beta-d\alpha^4}{a}$$

$$=\frac{-5b^3-5abc+a^2d}{a^7}.$$

&c. = &c.

Substitute the values for α, β, γ, δ, &c. then

$$y=\frac{x}{a}-\frac{bx^2}{a^3}+\frac{(2b^2-ac)x^3}{a^5}-\frac{(5b^3-5abc+a^2d)x^4}{a^7}+\text{\&c.};$$

and if $a=1$, or $x=y+by^2+cy^3+dy^4+$ &c. then $y=x-bx^2+(2b^2-c)x^3-(5b^3-5bc+d)x^4+$ &c. (A).

169. In the following chapter it will be shewn, that if x be the logarithm of the number $1+n$, $x=n-\frac{1}{2}n^2+\frac{1}{3}n^3-\frac{1}{4}n^4+$ &c.; suppose therefore it was required to find the *number* in terms of the *logarithm*, i.e. to find n in terms of x, then comparing the equation $x=n-\frac{1}{2}n^2+\frac{1}{3}n^3-\frac{1}{4}n^4+$ &c. with the equation $x=y+by^2+cy^3+dy^4+$ &c. and substituting n for y in the equation (A), we should have $n=x-bx^2+(2b^2-c)x^3-(5b^3-5bc+d)x^4+$ &c. where $b=-\frac{1}{2}$, $c=\frac{1}{3}$, $d=-\frac{1}{4}$; &c.;

$$\text{Hence} -b=\frac{1}{2},$$

$$2b^2-c=\frac{1}{2}-\frac{1}{3}=\frac{1}{2.3}$$

$$-(5b^3-5bc+d)=\frac{5}{8}-\frac{5}{6}+\frac{1}{4}=\frac{15-20+6}{24}=\frac{1}{2.3.4},$$

&c. = &c.

$$\therefore n=x+\frac{x^2}{2}+\frac{x^3}{2.3}+\frac{x^4}{2.3.4}+\text{\&c.};$$

$$\text{and } 1+n=1+x+\frac{x^2}{2}+\frac{x^3}{2.3}+\frac{x^4}{2.3.4}+\text{\&c.}$$

CHAP. X.

ON LOGARITHMS,

AND SUBJECTS CONNECTED WITH THEM.

XLVII.

Definition and Properties of Logarithms.

170. In the two following series of quantities, a^x, $a^{x'}$, $a^{x''}$, $a^{x'''}$, &c. (A); x, x', x'', x''', &c. (B); where a is some given number greater or less than unity, and x, x', x'', x''', &c. any variable quantities whatever, the several terms of the series (B) are called the *logarithms* of the several terms corresponding to them in the series (A). Thus if $a^x=y$, $a^{x'}=y'$, $a^{x''}=y''$, &c. then $x=\log. y$; $x'=\log. y'$; $x''=\log. y''$; &c.

171. In adapting the series (A) to the numbers 1, 2, 3, 4, 5, 6, &c. the given number a must be *greater* than unity, the first index x must be equal to 0, and the several indices x', x'', x''', &c. must keep continually increasing. For in this case, since (by Art. 68.) $a^0=1$, this series will increase from 1 to infinity; and by properly adjusting the values of x', x'', x''', &c. it is evident that the several quantities $a^{x'}$, $a^{x''}$, $a^{x'''}$, &c. may be made to coincide with the numbers 2, 3, 4, 5, 6, &c. For instance, let $a=10$; then (since $10^0=1$ and $10^1=10$), the indices of 10 which would give $10^{x'}$, $10^{x''}$, $10^{x'''}$, &c. equal to the numbers 2, 3, 4, 5, &c. must be fractions between 0 and 1.

Take for example the number 5. Now $10^{\frac{2}{3}}=\sqrt[3]{10^2}=\sqrt[3]{100}$ $=4.64$; from which we infer, that a fraction (x') somewhat greater than $\frac{2}{3}$ ($=.666666$, &c.) being made the index of 10, would give $10^{x'}=5$; this fraction is found by calculation to be .6989700; hence $10^{.6989700}=5$; i.e. when $a=10$, the logarithm of 5 is .6989700.

172. From hence it appears that the logarithm of any given number will depend upon the value of a, and that different systems of logarithms would be formed by assuming it equal to different numbers, but that (since $a^0=1$) in every system the logarithm of *one* would be 0. This constant quantity a, from whose powers the natural numbers are formed, is called the *base* of the system to which it belongs. But before we proceed to calculate a system of logarithms, it will be proper to explain some of their *properties*.

173. Let N and n be any two numbers belonging to the series (A); let N (for instance) $=a^x$, and $n=a^{x''''}$; then $Nn=a^x\times a^{x''''}=a^{x+x''''}$; but by Art. 170, the *logarithm* of $a^{x+x''''}$ is $x+x''''$, ∴ the logarithm of $Nn=x+x''''=\text{log.}\ a^x+\text{log.}\ a^{x''''}=\text{log.}\ N+\text{log.}\ n$. In the same manner, if n, n', n'', n''', &c. be any set of numbers belonging to the series (A), it might be shewn that the logarithm of $nn'n''n'''$, &c. $=\text{log.}\ n+\text{log.}\ n'+\text{log.}\ n''+\text{log.}\ n'''+$ &c.; from which we infer that "the logarithm of the *product* of any number of factors is equal to the *sum of their logarithms*."

174. Again, $\frac{N}{n}=\frac{a^x}{a^{x''''}}=a^{x-x''''}$; but the logarithm of $a^{x-x''''}$ $=x-x''''$; ∴ the logarithm of $\frac{N}{n}=x-x''''=\text{log.}\ a^x-\text{log.}\ a^{x''''}$ $=\text{log.}\ N-\text{log.}\ n$; from hence it appears that "the logarithm of "the *quotient* of any two numbers is equal to the *difference of "their logarithms*; and that the logarithm of a *fraction* $\left(\frac{N}{n}\right)$

"is equal to the logarithm of its *numerator* minus the logarithm "of its denominator." If N be less than n, then $\log. N - \log. n$ is *negative*; consequently the logarithms of all *proper* fractions are negative quantities.

175. Let $N=a^x$ be raised to the mth power, then $N^m=a^{mx}$; but the logarithm of $a^{mx}=mx$; hence the logarithm of $N=mx$ $=m.\log.a^x=m.\log.N$; for the same reason, since $\sqrt[m]{N}=N^{\frac{1}{m}}$ $=a^{\frac{x}{m}}$, the logarithm of $\sqrt[m]{N}=\frac{x}{m}=\frac{\log. N}{m}$; from which we infer that "the logarithm of the mth power of any number is "found by *multiplying* its logarithm by m; and of the mth "root of any number, by *dividing* its logarithm by m."

176. If the series (A) consists of quantities of the form a^x, a^{2x}, a^{3x}, a^{4x}, &c......a^{nx}, then the corresponding terms of the series (B) are x, $2x$, $3x$, $4x$, &c.....nx; i.e. "if a series of "quantities be in *geometrical* progression, their logarithms will "be in *arithmetical* progression."

XLVIII.

On the Method of finding the Logarithm of any given Number.

177. Let a^x express *generally* any term of the series (A), and let the corresponding number be N, then $a^x=N$. To find the logarithm of N, is merely to solve the equation $a^x=N$, where x is the unknown quantity; for this purpose let $a=1+b$, and $N=1+n$; then $(1+b)^x=1+n$; extract the root y of each side of this equation, and we have $(1+b)^{\frac{x}{y}}=(1+n)^{\frac{1}{y}}$; ∴ by expansion,

$$1+\frac{x}{y}(b)+\frac{x}{y}\left(\frac{x}{y}-1\right)\left(\frac{b^2}{2}\right)+\frac{x}{y}\left(\frac{x}{y}-1\right)\left(\frac{x}{y}-2\right)\left(\frac{b^3}{2.3}\right)+\text{\&c.}$$

$$=1+\frac{1}{y}(n)+\frac{1}{y}\left(\frac{1}{y}-1\right)\left(\frac{n^2}{2}\right)+\frac{1}{y}\left(\frac{1}{y}-1\right)\left(\frac{1}{y}-2\right)\left(\frac{n^3}{2.3}\right)+\text{\&c.}$$

Now suppose y to be indefinitely great, then will $\frac{x}{y}$ and $\frac{1}{y}$ vanish with respect to -1, -2, &c. so that $\frac{x}{y}-1$ and $\frac{1}{y}-1$ will be each equal to -1; $\frac{x}{y}-2, \frac{1}{y}-2$, each equal to -2; &c. &c.; hence, rejecting 1 from each side of the equation, we have

$$\frac{x}{y}(b-\tfrac{1}{2}b^2+\tfrac{1}{3}b^3-\tfrac{1}{4}b^4+\text{\&c.})=\frac{1}{y}(n-\tfrac{1}{2}n^2+\tfrac{1}{3}n^3-\tfrac{1}{4}n^4+\text{\&c.})$$

$$\therefore\ x(=\log.\ \overline{1+n})=\frac{n-\tfrac{1}{2}n^2+\tfrac{1}{3}n^3-\tfrac{1}{4}n^4+\text{\&c.}}{b-\tfrac{1}{2}b^2+\tfrac{1}{3}b^3-\tfrac{1}{4}b^4+\text{\&c.}}=$$

$$\left(\text{since}\begin{array}{l} n=N-1 \\ b=a-1 \end{array}\right)\frac{(N-1)-\tfrac{1}{2}(N-1)^2+\tfrac{1}{3}(N-1)^3-\text{\&c.}}{(a-1)-\tfrac{1}{2}(a-1)^2+\tfrac{1}{3}(a-1)^3-\text{\&c.}}$$

178. Let $(a-1)-\frac{1}{2}(a-1)^2+\frac{1}{3}(a-1)^3-\text{\&c.}=M$. This quantity ($M$), which depends upon the value of the base a, is called the *Modulus* of the particular system of logarithms to which it belongs. But before we proceed to the calculation of the logarithms belonging to any particular system, it will be proper to shew the manner in which the value of x, as deduced in the last article, may be expressed in terms of a *converging* series; for the series $n-\frac{1}{2}n^2+\frac{1}{3}n^3-\frac{1}{4}n^4+\frac{1}{5}n^5-\text{\&c.}$ will not converge, when n is any whole number greater than unity. This may be effected by means of the following process, in which M is substituted for the denominator of the fraction; thus,

$$\text{Log.}\ (1+n)=\frac{1}{M}\left(n-\frac{1}{2}n^2+\frac{1}{3}n^3-\frac{1}{4}n^4+\frac{1}{5}n^5-\text{\&c.}\right)[C].$$

For n put $-n$,

$$\text{then}\ \log.\ (1-n)=\frac{1}{M}\left(-n-\frac{1}{2}n^2-\frac{1}{3}n^3-\frac{1}{4}n^4-\frac{1}{5}n^5-\text{\&c.}\right)[D].$$

Subtract [D] from [C], then $\log.\ (1+n)-\log.\ (1-n)=$

$$\log.\ \frac{1+n}{1-n}=\frac{2}{M}\left(n+\frac{1}{3}n^3+\frac{1}{5}n^5+\text{\&c.}\right).$$

Now let $n=\frac{1}{N-1}$, then $\frac{1+n}{1-n}=\frac{1+\frac{1}{N-1}}{1-\frac{1}{N-1}}=\frac{N}{N-2}$.

Hence, by substitution,

$$\log.\frac{N}{N-2}=\frac{2}{M}\left(\frac{1}{N-1}+\frac{1}{3(N-1)^2}+\frac{1}{5(N-1)^5}+\&c.\right), \text{ or}$$

$$\log.N-\log.(N-2)=\frac{2}{M}\left(\frac{1}{N-1}+\frac{1}{3(N-1)^3}+\frac{1}{5(N-1)^5}+\&c.\right);$$

$$\therefore \log.N=\frac{2}{M}\left(\frac{1}{N-1}+\frac{1}{3(N-1)^3}+\frac{1}{5(N-1)^5}+\&c.\right)+\log.(N-2);$$

which is a series that converges very fast, when N is any number greater than 2.

XLIX.

On the Method of constructing a Table of Logarithms.

179. Before we can proceed to calculate a table of logarithms, some value must be assigned to M. Since a may be arbitrarily assumed, let it be so assumed that $(a-1)-\frac{1}{2}(a-1)^2+\frac{1}{3}(a-1)^3-\&c.$ or M shall be equal to 1. As N must be greater than 2 (Art. 178), we cannot immediately apply the series for log. N to the finding of *log*. 2; but the logarithm of 2 may be found in the following manner; let $N=4$, then

$$\text{Log}.\ 4=\log.\ 2^2=2\log.\ 2=(\text{for } M=1)2\left(\frac{1}{3}+\frac{1}{3^4}+\frac{1}{5.3^5}+\&c.\right)$$
$$+\log.\ 2:$$

$$\therefore 2\log.2-\log.\ 2, \text{ or } \log.\ 2=2\left(\frac{1}{3}+\frac{1}{3^4}+\frac{1}{5.3^5}+\&c.\right).$$

Having the logarithm of 2, a table of logarithms, in which $M=1$, may be constructed in the following manner:

Log.
1 $=0$
2 $=2\left(\frac{1}{3}+\frac{1}{3^4}+\frac{1}{5.3^5}+\&c.\text{ to 7 terms}\right)$ . . . . $=0.6931471$
3 $=2\left(\frac{1}{2}+\frac{1}{3.2^3}+\frac{1}{5.2^5}+\&c.\text{ to 10 terms}\right)+\log.\ 1(0)$. $=1.0986122$
4 $=2\log.\ 2$ . . . . $=1.3862942$
5 $=2\left(\frac{1}{4}+\frac{1}{3.4^3}+\frac{1}{5.4^5}+\&c.\text{ to 6 terms}\right)+\log.\ 3$ . . . $=1.6094378$
6 $=\log.\ 3+\log.\ 2$ . . . . $=1.7917593$
7 $=2\left(\frac{1}{6}+\frac{1}{3.6^3}+\frac{1}{5.6^5}+\frac{1}{7.6^7}\right)+\log.\ 5$ . . . . $=1.9458999$
8 $=\log.\ 4+\log.\ 2$, or $\log.\ 2^3=3\log.\ 2$ . . . . $=2.0794413$
$=\log.\ 3^2=2\log.\ 3$ . . . . $=2.1972244$
10 $=\log.\ 5+\log.\ 2$ . . . . $=2.3025849$
&c. $=$&c. . . . . $=$&c.

In this table, the number of terms mentioned in each case are sufficient to make the logarithms true to 7 places of decimals. The logarithms here calculated are called *Napier's* logarithms, from the name of their celebrated inventor; but they are at present more commonly known by the name of *Hyperbolic* logarithms, on account of their application to the quadrature of the equilateral hyperbola.

180. To find the value of the base a of this system, we must have recourse to the series [C] in Art. 178, in which, making $\log.\ (1+n)$ or $\log.\ N=l$ and $M=1$, we have, $l=n-\frac{1}{2}n^2+\frac{1}{3}n^3-\frac{1}{4}n^4+\&c.$ and *reverting* this series by Art. 169, (putting l for x) we obtain, $1+n$ or $N=1+l+\frac{1}{2}l^2+\frac{1}{2.3}(l^3)+\frac{1}{2.3.4}(l^4)+\&c.$ Let $l=1$, then the number whose logarithm is 1 (i.e. a) $=1+1+\frac{1}{2}+\frac{1}{2.3}+\frac{1}{2.3.4}+\&c.=2.7182818$. The *base* of this system of logarithms, therefore, is 2.7182818, and the logarithms of the several powers of a, viz. a, a^2, a^3, &c.

being 1, 2, 3, &c. we might have interposed in the preceding table,

log. 2.7182818 $=1.0000000$,
log. 7.3742037 being the *square* of 2.7182818) $=2.0000000$,
&c. $=$&c.

Hence, in this system, the numbers whose logarithms are 1, 2, 3, &c. are *decimal* numbers; and from this circumstance they are very inconvenient for ordinary arithmetical calculations.

181. In the *common* system of logarithms, the base $a=10$, $\therefore a^2=100, a^3=1000, a^4=10000$, &c.; hence, if the *numbers* be 1, 10, 100, 1000, 10000, &c. their *logarithms* are 0, 1, 2, 3, 4, &c.; and to construct the table for the intermediate numbers, we must find the value of M when $a=10$. Now, in the system whose modulus is M, log. $N=\frac{1}{M}(n-\frac{1}{2}n^2+\frac{1}{3}n^3-\frac{1}{4}n^4+\&c.)$; let $M=1$, then *hyp. log.* $N=n-\frac{1}{2}n^2+\frac{1}{3}n^3-\frac{1}{4}n^4+\&c.$; hence, in the system whose modulus is M, log. $N=\frac{\text{hyp. log. } N}{M}$; let $N=10$, then log. $10=\frac{\text{hyp. log. } 10}{M}$; but in the common system, log. $10=1$, $\therefore 1=\frac{\text{hyp. log. } 10}{M}$, and $M=$hyp. log. $10=$(by table in Art. 179.) 2.3025849. We are thus enabled to find the value of $\frac{2}{M}$, which is the common multiplier of the series at the end of Art. 178, for $\frac{2}{M}=\frac{2}{2.3025849}$ $=.86858896$. To construct a table of *common* logarithms, therefore, we have only to substitute for $\frac{2}{M}$ its value thus found, in the general series $\frac{2}{M}\left(\frac{1}{N-1}+\frac{1}{3(N-1)^3}+\frac{1}{5(N-1)^5}+\&c.\right)+\log.(N-2)$. Thus,

Log.

$2 = .86858896\left(\frac{1}{3}+\frac{1}{3^4}+\frac{1}{5{\cdot}3^5}+\text{\&c. to 7 terms}\right) = 0.3010300$

$3 = .86858896\left(\frac{1}{2}+\frac{1}{3.2^3}+\frac{1}{5{\cdot}3^5}+\text{\&c. to 10 terms}\right) = 0.4771213$

$4 = 2 \log. 2 \ldots\ldots\ldots\ldots = 0.6020600$

$5 = \log. \frac{10}{2} = \log. 10 - \log. 2 = 1 - \log. 2 \ldots\ldots = 0.6989700$

$6 = \log. 3 + \log. 2 \ldots\ldots\ldots\ldots = 0.7781513$

$7 = .86858896\left(\frac{1}{6}+\frac{1}{3.6^3}+\frac{1}{5{\cdot}6^5}+\frac{1}{7{\cdot}6^7}\right)+\log. 5 \ldots = 0.8450980$

$8 = \log. 2^3 = 3 \log. 2 \ldots\ldots\ldots\ldots = 0.9030900$

$9 = \log. 3^2 = 2 \log. 3 \ldots\ldots\ldots\ldots = 0.9542426$

$10 = \ldots\ldots\ldots\ldots\ldots\ldots = 1.0000000$

$11 = 86858896\left(\frac{1}{10}+\frac{1}{3{\cdot}10^3}+\frac{1}{5.10^5}\right)+\log. 9 \ldots = 1.0413927$

$12 = \log. 2 + \log. 4 \ldots\ldots\ldots\ldots = 1.0791812$

$13 = .86858896\left(\frac{1}{12}+\frac{1}{3{\cdot}12^3}+\frac{1}{5.12^5}\right)+\log. 11 \ldots = 1.1139434$

$14 = \log. 7 + \log. 2 \ldots\ldots\ldots\ldots = 1.1461280$

$15 = \log. 5 + \log. 3 \ldots\ldots\ldots\ldots = 1.1760913$

$16 = \log. 4^2 = 2 \log. 4 \ldots\ldots\ldots\ldots = 1.2041200$

$17 = 86858896\left(\frac{1}{16}+\frac{1}{3.16^3}+\frac{1}{5{\cdot}16^5}\right)+\log. 15 \ldots = 1.2304489$

$18 = \log. 9 + \log. 2 \ldots\ldots\ldots\ldots = 1.2552725$

$19 = .86858896\left(\frac{1}{18}+\frac{1}{3{\cdot}18^3}+\frac{1}{5.18^5}\right)+\log. 17 \ldots = 1.2787536$

$20 = \log. 10 + \log. 2 \ldots\ldots\ldots\ldots = 1.3010300$

$21 = \log. 7 + \log. 3 \ldots\ldots\ldots\ldots = 1.3222193$

$22 = \log. 11 + \log. 2 \ldots\ldots\ldots\ldots = 1.3424227$

$23 = .86858896\left(\frac{1}{22}+\frac{1}{3{\cdot}22^3}+\frac{1}{5{\cdot}22^5}\right)+\log. 21 \ldots = 1.3617278$

The next number which requires calculation by means of the series, is 29; and from this number to 400 *inclusive*, *two* terms of the series are sufficient to make the logarithms true to 7 places of decimals. After 400, *one* term is sufficient; thus log. 401 $= \frac{.86858896}{400} + \log. 399 = .0021714724 + 2.6009729 =$ 2.6031444 (very nearly); and in this manner the table might be continued with great facility to any extent, by means of the logarithms previously calculated. For the most expeditious manner of dividing the number .86858896 by the denominators of the several fractions composing the series, and for the manner of using logarithmic tables, the reader is referred to the Preface annexed to Dr. Hutton's *Tables*.

182. Since log. $1=0$, log. $10=1$; log. $100=2$; log. $1000 =3$, &c., it follows that the logarithms of all numbers between 1 and 10 will be some decimal number less than unity; between 10 and 100, some decimal number between 1 and 2; between 100 and 1000, some decimal number between 2 and 3; &c.&c. The *whole number* annexed to the decimal is called the *index* or *characteristic* of the logarithm; and consequently for all numbers between 10 and 100 the index is 1; between 100 and 1000, the index is 2; between 1000 and 10000, the index is 3; &c. &c. From the circumstance of log. $10=1$, it also follows that the logarithms of all numbers in *decuple* proportion involve the same decimal number, and differ only by their *index*.

Thus, Log. 1132 . . . . $=3.0538464$.

$$\text{Log. } 113.2 = \text{log. } \frac{1132}{10} = \text{log. } 1132 - 1 = 2.0538464.$$

$$\text{Log. } 11.32 = \text{log. } \frac{113.2}{10} = \text{log. } 113.2 - 1 = 1.0538464.$$

$$\text{Log. } 1.132 = \text{log. } \frac{11.32}{10} = \text{log. } 11.32 - 1 = 0.0538464.$$

$$\text{Log. } 11.32 = \text{log. } \frac{1.132}{10} = \text{log. } .1132 - 1 = \underline{1}.0538464.$$

$$\text{Log. } .01132 = \text{log. } \frac{.1132}{10} = \text{log. } .1132 - 1 = \underline{2}.0538464.$$

$$\text{Log. } .001132 = \text{log. } \frac{.01132}{10} = \text{log.} .01132 - 1 = \underline{3}.0538464;$$

where the negative sign placed under the index of the last three logarithms shews that it is the logarithm of a *fraction*. To find the *real* logarithm of the number, the decimal part must be subtracted from the index, and the negative sign annexed to the result.

183. The foregoing property, belonging to that particular system of logarithms arising out of the supposition of the base $a=10$, is not only of great practical utility in their application to arithmetical purposes, but also very much facilitates the construction and use of the tables founded upon that system. Since the same decimal logarithm always applies to a number consisting of the same digits, it follows that in the construction of a table of common logarithms it is only necessary to register the digits of the number and the decimal logarithm in parallel columns; for the *index* of the logarithm may always be determined from the actual value of the number, and *vice versa*, the actual value of the number may always be determined from the index of the logarithm. For instance, in the common tables where the logarithms are registered for all numbers consisting of five figures, the decimal logarithm belonging to the number 98637 is .9940398; if this number be a *whole* number, then since it lies between 10000 and 100000 (the index of which is 4) we know that its logarithm is 4.9940398; if a decimal point be placed before the last figure, then the value of the number is 9863.7 (which lies between 1000 and 10000) and therefore its logarithm is 3.9940398; if a decimal point be placed before the last figure but one, then the number is 986.37, and its logarithm 2.9940398; &c. &c. On the other hand, if the logarithm 1.9940398 was given to find the corresponding number, then since the decimal part of it belongs to the digits 98637, and since from the index of the logarithm we know that the number lies between 10 and 100, the figures 98637 must be so pointed as to make the number lie between 10 and 100, viz. 98.637; &c. &c. The utility of this system was so obvious, that

the tables for ordinary purposes were founded upon it very soon after the invention of logarithms. For further information upon this subject, the Reader is referred to the Appendix of Mr. WOODHOUSE's *Trigonometry*, Ed. 1809.

L.

On the application of Logarithms to Complex Arithmetical Operations.

184. Logarithms are of no very great use in the ordinary operations of multiplying or dividing one number by another; it is in the raising of powers, and the extraction of roots, and in their application to complicated numerical expressions, that their utility most plainly appears.

EXAMPLE 1.

Find the 5th root of 2593.

By Art. 175, the logarithm of the 5th root of 2593 $= \frac{\text{log. } 2593}{5} = \frac{3.4138025}{5} = .6827605 = \text{log. } 4.8168$; $\therefore$ the 5th root of $2593 = 4.8168$.

EXAM. 2.

Find the value of the fraction $\frac{2^{20} \times 3^7 \times 2.013}{17 \times 9350}$.

By Art. 174, the logarithm of this fraction is equal to the log. of its numerator *minus* log. of its denominator.

By Arts. 173, 175, $\text{log.} 2^{20} \times 3^7 \times 2{,}013 = 20 \text{log.} 2 + 7 . \text{log.} 3 . + \text{log.} 2.013$.

. . . . and, $\text{log.} 17 \times 9350 = \text{log. } 17 + \text{log. } 9350$.

Now $20 \times \text{log.} 2 = 6.0206000$ . . $\text{log. } 17 = 1.2304489$.

$7 \times \text{log.} 3 = 3.3398491$ . . $\text{log. } 9350 = 3.9708116$.

$\text{log.} 2.013 = 0.3038438$

By addition $= 9.6642929$ (*A*). 5.2012605 (*B*).

Subtract (*B*) from (*A*), and we have 4.4630324, which is the logarithm of 29042, the number required.

EXAM. 3.

Find the value of $\sqrt[5]{\dfrac{(317)^2 \times \sqrt{3}\sqrt[3]{5}}{251}}$

Call the *numerator* of this fraction (N), and its *denominator* (n);

Then, by Arts. 174, 175, log. of $\sqrt[5]{\dfrac{N}{n}} = \dfrac{\text{log.}\,N - \text{log.}\,n}{5}$.

$$\begin{aligned} \text{Now log.}\,(317)^2 &= 2 \times \text{log.}\,317 = 5.0021186. \\ \text{log.}\ \sqrt{3} &= \tfrac{1}{2} \times \text{log.}\ 3 = 0.2385606. \\ \text{log.}\ \sqrt[3]{5} &= \tfrac{1}{3} \times \text{log.}\ 5 = \underline{0.2329900.} \\ &\ 5.4736692 = \text{log.}\,N. \\ \text{log.}\,251 &= \underline{2.3996737;} \\ \therefore &\ \ \ 3.0739955 = \text{log.}\,N - \text{log.}\,n. \end{aligned}$$

Hence $\dfrac{\text{log.}\,N - \text{log.}\,n}{5} = \dfrac{3.0739955}{5} = 0.6147991$, which is the logarithm of 4.119 for the number required.

EXAM. 4.

Find a fourth proportional to the 6th power of 9, the 4th power of 7, and the 5th power of 5.

Let x = the number required, then $9^6 : 7^4 :: 5^5 : x = \dfrac{7^4 \times 5^5}{9^6}$;

$\therefore$ log. x = 4 log. 7 + 5 log. 5 − 6 log. 9 = 3.3803920 + 3.4948500 − 5.7254550 = 1.1497870 = log. 14.118; hence x = 14.118.

Ex. 5. Find the value of $\dfrac{31 \times 33 \times 255 \times 315.}{35 \times 357}$

ANSWER, 6576.4.

Ex. 6. Divide the 20th power of 2 by the 12th power of 3.

ANSW. 1.973.

Ex. 7. Find a *third proportional* to $\sqrt[5]{117}$ and $\sqrt[3]{137}$.

ANSW. 10.252.

Ex. 8. Find the value of $\sqrt[7]{\dfrac{935 \times \sqrt{14} \times \sqrt[3]{100}.}{\sqrt[4]{2}.}}$

ANSW. 3.897.

LI.

On the Summation of Geometric Series.

185. Logarithms are found very useful in ascertaining the value of S in the equation $S=\frac{ar^n-a}{r-1}$ or $\frac{a-ar^n}{1-r}$, where a is the *first* term, r the *common ratio,* and n the number of terms of a geometric series.

EXAMPLE I.

Find the sum of 20 terms of the series $1, \frac{3}{2}, \frac{9}{4}, \frac{27}{8}$, &c.

$$\left.\begin{array}{r}\text{Here } a=1,\\ r=\frac{3}{2},\\ n=20;\end{array}\right\}\therefore\ S=\frac{ar^n-a}{r-1}=\frac{1\times\left(\frac{3}{2}\right)^{20}-1}{\frac{3}{2}-1}=2\times\left(\left(\frac{3}{2}\right)^{20}-1.\right)$$

$$\begin{aligned}\text{Now log. }\left(\frac{3}{2}\right)^{20}&=20\times\text{log. }\frac{3}{2}.\\ &=20\times(\text{log. }3-\text{log. }2.)\\ &=3.5218260=\text{log. }3325.262;\end{aligned}$$

$$\therefore\left(\frac{3}{2}\right)^{20}=3325.262.$$

Hence $S=2\times\left(\left(\frac{3}{2}\right)^{20}-1\right)=2\times3324.262=6648.524.$

Ex. 2.

Find the sum of 10 terms of the series $1, \frac{5}{6}, \frac{25}{36}, \frac{125}{216}$, &c.

$$\left.\begin{array}{r}\text{Here } a=1,\\ r=\frac{5}{6},\\ n=10;\end{array}\right\}\therefore S=\frac{a-ar^n}{1-r}=\frac{1-1\times\left(\frac{5}{6}\right)^{10}}{1-\frac{5}{6}}=6\times\left(1-\left(\frac{5}{6}\right)^{10}.\right)$$

$$\begin{aligned}\text{Now log. }\left(\frac{5}{6}\right)^{10}&=10\times\text{log. }\frac{5}{6}\\ &=10\times(\text{log. }5-\text{log. }6.)\end{aligned}$$

$$=10\times-.0791813.$$
$$=-.7918130.$$
$$=.2081870-1.0000000.$$
$$=\text{log. }1.6150-\text{log. }10.$$
$$\therefore\ \left(\frac{5}{6}\right)^{10}=\frac{1.6150}{10}=1615.$$

$$\text{Hence } S=6\left(1-\left(\frac{5}{6}\right)^{10}\right)=6(1-.1615)=5.031.$$

186. If the *sum* of the series, the *common ratio*, and the *first term* be given; the *number of terms* may be found thus;

$$\text{Since } rS-S=ar^{n}-a;$$
$$\text{By } \textit{transposition},\ ar^{n}=rS-S+a,$$
$$\text{and } r^{n}=\frac{rS-S+a}{a};$$
$$\therefore \text{log. } r^{n} \text{ or } n\times\text{log. } r=\text{log. }(rS-S+a)-\text{log. } a.$$
$$\text{Hence } n=\frac{\text{log. }(rS-S+a)-\text{log. } a}{\text{log. } r}.$$

Ex. 3.

The *sum* of a geometric series is 6560, its *first term* 2, and *common ratio* 3. What is the number of terms?

$$\text{Here }\left.\begin{array}{l}S=6560,\\ a=2,\\ r=3;\end{array}\right\}\quad \therefore\ n=\frac{\text{log. }(rS-S+a)-\text{log. } a}{\text{log. } r}.$$
$$=\frac{\text{log. }13122-\text{log. }2}{\text{log. }3}.$$
$$=\frac{3.8169704}{.4771213}=8.$$

Ex. 4. A servant agreed to serve his master for one year (13 months), at the rate of sixpence for the *first* month, a shilling for the *second*, two shillings for the *third*, and so on; What had he to receive at the end of the year?

ANSWER, £.204. 15*s*. 6*d*.

Ex. 5. Find the sum of 11 terms of the series, $1, \frac{5}{4}, \frac{25}{16}$, &c.

Answ. 42.568.

Ex. 6. The *sum* of a geometric series is 1023, the *first term* 1, and *common ratio* 2; Find the *number of terms.*

Answ. 10.

Ex. 7. A person undertakes a journey of 364 miles, going *one* mile the *first* day, *three* the *second*, *nine* the *third*, and so on; When will he arrive at his journey's end?

Answ. in 6 days.

LII.

On Compound Interest.

Let (P) be the *principal*, or sum put out to *compound* interest; (r) the fraction which expresses the *rate* of interest per cent.; (A) the *amount* at the end of (n) years; Then the following Theorems may be established, by means of logarithms.

Theorem 1.

187. "*Log.* $A = \log. P + n \times \log. (1+r)$."

For since £.1, at the end of the *first* year, becomes $1+r$, and that the amount is increased *each* year in the same ratio, we have, by the rule of proportion,

$1 : 1+r :: P : P(1+r) =$ amount of P at end of *first* year.
$1 : 1+r :: P(1+r) : P(1+r)^2 =$ - - - - - - - - - *second* year.
$1 : 1+r :: P(1+r)^2 : P(1+r)^3 =$ - - - - - - - - - *third* year.
&c. &c.

So that, at the end of n years, the amount is $P\,(1+r)^n$.

Hence $A = P(1+r)^n$;

and, taking the *logarithm*, $\log. A = \log. P + n \times \log. (1+r)$.

From which we deduce,

$$\text{Log. } P = \log. A - n \times \log. (1+r).$$

$$\text{Log. } (1+r) = \frac{\log. A - \log. P}{n},$$

and

$$\text{and } n=\frac{\log. A-\log. P}{\log. (1+r)}.$$

Any *three* of the quantities A, P, r, n, being given, the *fourth* may therefore be found.

THEOR. 2.

188. "Let $A=mP$, then $n=\frac{\log. m}{\log. (1+r)}$."

For, in this case, $mP=P(1+r)^n$.

Divide by P, then $m=(1+r)^n$.

Take the logarithm, $\log. m=n\times\log. (1+r)$; $\therefore n=\frac{\log. m}{\log. (1+r)}$.

By means of this Theorem, we ascertain the period or number of years in which a sum of money would *double*, *treble*, &c. or amount to m times itself, when put out at compound interest, at r rate per cent.

THEOR. 3.

189. "*Suppose the interest to be paid half yearly, and at the* "*same time converted into principal, then will* $\log. A=\log. P$ "$+2n\times\log.(1+\frac{1}{2}r)$."

For in this case, $2n$ must be substituted for n, and $\frac{1}{2}r$ for r.

Hence, at the end of n years, $A=P(1+\frac{1}{2}r)^{2n}$;

and, taking the logarithm, $\log. A=\log. P+2n\times\log. (1+\frac{1}{2}r)$.

THEOR. 4.

190. "*Suppose now, that besides the interest being converted* "*into principal at the end of every year, the sum* P *is at the* "*same time invested in capital; then the amount* (A), *at the end of* n *years, will be* $\frac{PR(R^n-1)}{R-1}$ (If $R=1+r$)."

In this case, the principal (P) is put out for n, $n-1$, $n-2$, &c. years, in succession; the amount therefore is the sum of the several amounts of (P) put out for n, $n-1$, $n-2$, &c. years;

$\therefore A=P$

$$\begin{aligned}\therefore A&=P(1+r)^n+P(1+r)^{n-1}+P(1+r)^{n-2}+\&\text{c.}P(1+r).\\ &=(\text{if } 1+r=R)PR^n+PR^{n-1}+PR^{n-2}+\&\text{c.}\ \ldots\ PR.\\ &=P(R^n+R^{n-1}+R^{n-2}\&\text{c.}\ \ldots\ldots\ R).\\ &=P\times(\text{Geom.Prog.}\textit{ first term } R^n, \textit{common ratio } R)=\frac{P(R^{n+1}-R)}{R-1}.\\ &=\frac{PR(R^n-1)}{R-1}.\end{aligned}$$

EXAMPLE 1.

What would be the amount of 200*l*. placed out for 7 years, at 4 per cent. compound interest?

$$\left.\begin{array}{r}\text{Here } P=200,\\ r=\frac{1}{25},\\ 1+r=1+\frac{1}{25},\\ =1.04,\\ n=7;\end{array}\right\}\begin{array}{l}\therefore \text{by Th.1.log.}A=\text{log.}P+n\times\text{log.}(1+r).\\ \qquad=\text{log.}200+7\times\text{log.}1.04.\\ \qquad=2.4202631.\\ \qquad=\text{log. }263.18.\\ \text{Hence, } A=263l.\ 3s.\ 7\tfrac{1}{5}d.\end{array}$$

Ex. 2.

How much money must be placed out at compound interest, to amount to 500*l*. in 12 years, at 5 per cent?

$$\left.\begin{array}{r}\text{Here } A=500,\\ r=\frac{1}{20},\\ 1+r=1+\frac{1}{20},\\ =1.05,\\ n=12.\end{array}\right\}\begin{array}{l}\text{By Th. 1. log. }P=\text{log.}A-n\times\text{log.}(1+r).\\ \qquad=\text{log.}500-12\times\text{log.}1.05.\\ \qquad=2.4446984.\\ \qquad=\text{log. }278.41.\\ \text{Hence, } P=278l.\ 8s.\ 2\tfrac{2}{5}d.\end{array}$$

Ex. 3.

At what rate of interest must 400*l*. be placed out, that it may amount to 569*l*. 6*s*. 8*d*. in 9 years, at compound interest?

$$\left.\begin{array}{r}l.\quad s.\ \ d.,\\ \text{Here } A=569.\ 6.\ 8.\\ P=400,\\ n=9.\end{array}\right\}\begin{array}{l}\text{By Th.1. log.}(1+r)=\dfrac{\text{log.}A-\text{log.}P}{n}.\\ \qquad=\dfrac{\text{log.}589.33-\text{log.}400}{9}.\\ \qquad=.0703338.\\ \qquad=\text{log. }1.04=\text{log. }\left(1+\frac{1}{25}\right).\end{array}$$

$$\text{Hence } 1+r=1+\frac{1}{25};$$

$$\therefore r=\frac{1}{25}, \text{ or rate of interest 4 per cent.}$$

Ex. 4.

In how many years will 500*l.* amount to 900*l.*, at 5 per cent. compound interest?

$$\left.\begin{aligned} \text{Here } A&=900,\\ P&=500,\\ r&=\frac{1}{20},\\ 1+r&=1.05. \end{aligned}\right\} \begin{aligned} \text{By Theor. 1. } n&=\frac{\log. A-\log. P}{\log. 1+r}.\\ &=\frac{\log. 900-\log. 500}{\log. 1.05}.\\ &=\frac{.2552725}{.0211893}=12.04 \text{ years.} \end{aligned}$$

Ex. 5.

In what time will a sum of money *double* and *treble* itself at 5 per cent. compound interest?

By Theor. 2. $\left(\text{since } r=\frac{1}{20}\right)$.

$$\text{If } m=2, \text{ then time of } doubling=\frac{\log. 2}{\log. 1.05}=\frac{.3010300}{.0211893}=14.2 \text{ years.}$$

$$m=3 \;.\;.\;.\;.\;.\; \text{of } trebling=\frac{\log. 3}{\log. 1.05}=\frac{.4771213}{.0211893}=22.4 \text{ years.}$$

Ex. 6.

Supposing the interest to be paid *half yearly*, what will be the amount of 500*l.* in 8 years, at 5 per cent. compound interest?

$$\left.\begin{aligned} \text{Here } P&=500,\\ r&=\frac{1}{20},\\ 1+\tfrac{1}{2}r&=1.025,\\ n&=8. \end{aligned}\right\} \begin{aligned} \text{By Th. 3. } \log. A&=\log. P+2n\times\log. (1+\tfrac{1}{2}r).\\ &=\log. 500+16\times\log. (1.025).\\ &=2.8705524=\log. 742.25. \end{aligned}$$

Hence $A=742l.\ 5s.$

Ex. 7.

Suppose a person to place out annually 100*l.* for ten successive years, and suffer the whole to accumulate at the rate of 5 per cent. compound interest; What sum would he have to receive at the end of the tenth year?

$$\left.\begin{array}{r} \text{Here } P=100, \\ R=1.05, \\ n=10; \end{array}\right\} \quad \therefore \text{ by Theor. 4.} \quad A=\frac{PR(R^{n}-1)}{R-1}=\frac{105(\overline{1.05}|^{10}-1)}{.05}.$$

$$=2100(\overline{1.05}|^{10}-1).$$

$$\text{Now log. } (1.05)^{10}=10\times\text{log. } 1.05.$$

$$=.2118930.$$

$$=\text{log. } 1.6289;\ \therefore (1.05)^{10}-1=.6289.$$

$$\text{Hence } A=2100\times .6289.$$

$$=1320l.\ 13s.\ 9\tfrac{3}{5}d.$$

Examples for Practice.

Ex. 8. What would be the amount of 1000*l.* placed out at compound interest of 5 per cent. for 10 years?

Answer, 1628*l.* 16*s.*

Ex. 9. What sum must be placed out at compound interest, at 4 per cent., to amount to 2000*l.* in 15 years?

Answ. 1110*l.* 12*s.*

Ex. 10. At what rate of compound interest must 518*l.* 6*s.* be placed out, to amount to 600*l.* in 3 years?

Answ. 5 per cent.

Ex. 11. In how many years will 200*l.* amount to 318*l.* 16*s.*, at 6 per cent. compound interest?

Answ. 8 years.

Ex. 12. In how many years will a sum of money *double* itself, at 4 per cent. compound interest?

Answ. 17.6 years.

Ex. 13. Find the amount of 1200*l.* put out to compound interest at 6 per cent. for 10 years, the interest being converted into principal every *half* year.

ANS. 2167*l.* 6*s.*

Ex. 14. Suppose a person to place out annually the sum of 20*l.* for 40 successive years, and suffer the whole to accumulate, at the rate of 5 per cent. compound interest; What would he have to receive at the end of the 40 years?

ANS. 2536*l.* 16*s.*

LIII.

On the method of finding the Increase of Population in any Country, under given circumstances of Births and Mortality.

191. "Let (P) represent the population of a country at "any given period; $\left(\frac{1}{m}\right)$ the fractional part of the population "which die in a year (or ratio of mortality); $\left(\frac{1}{b}\right)$ the propor- "tion of births in a year; then, if (A) represents the state "of the population at the end of (n) years, $\log. A = \log. P$ "$+ n \times \log. \left(1 + \frac{m-b}{mb}\right)$."

The rate of increase of population in *one* year $= \frac{1}{b} - \frac{1}{m} = \frac{m-b}{mb}$;

$\therefore 1 : 1 + \frac{m-b}{mb} :: P : P\left(1 + \frac{m-b}{mb}\right) =$ state of population at the end of the *first* year.

But it is increased every year in the *same* proportion; $\therefore 1 : 1 + \frac{m-b}{mb} :: P\left(1 + \frac{m-b}{mb}\right) : P\left(1 + \frac{m-b}{mb}\right)^2 =$ state of the population at the end of the *second* year.

In the same manner we may prove, that the state of the population at the end of (n) years will be $P\left(1+\frac{m-b}{mb}\right)^n$.

$$\text{Hence } A=P\left(1+\frac{m-b}{mb}\right)^n;$$

$$\text{and log. } A=\text{log. } P.+n\times\text{log. }\left(1+\frac{m-b}{mb}\right).$$

From which we deduce,

$$\text{Log. } P=\text{log. } A-n\times\text{log. }\left(1+\frac{m-b}{mb}\right).$$

$$n=\frac{\text{log. } A-\text{log. } P}{\text{log. }\left(1+\frac{m-b}{mb}\right)}.$$

$$\text{log. }\left(1+\frac{m-b}{mb}\right)=\frac{\text{log. } A-\text{log. } P}{n}.$$

Of the quantities A, P, m, b, n, any *four* being given, the *fifth* may therefore be found.

EXAMPLE 1.

Suppose the population of Great Britain in the year 1800 to have been ten millions; that $\frac{1}{40}$th part *die* annually; that the births are to the deaths as 40 : 30; and that no emigration takes place during the present century; What will be the state of its new population in the year 1900?

$$\left.\begin{array}{r} \text{Here } p=10000000,\\ n=100,\\ m=40,\\ b=30;\ \text{and}\\ \therefore 1+\frac{m-b}{mb}=\frac{121}{120}.\end{array}\right\}\begin{array}{l} \text{Now log. } A=\text{log. } P+n\times\text{log. }\left(1+\frac{m-b}{mb}\right)\\ \qquad=\text{log. } 10000000+100\times\text{log. }\frac{121}{120}\\ \qquad=7.3604200\\ \qquad=\text{log. } 22930000.\\ \text{Hence } A=22930000.\end{array}$$

EXAM. 2.

Suppose the population of France, in the year 1792, to have been 27000000; the *ratio of mortality* during the 18th century to have been $\frac{1}{30}$th, and the *number of births* $\frac{1}{26}$th; What was the state of its population in the year 1700?

$$\left.\begin{aligned} \text{Here } A&=27000000,\\ n&=92,\\ m&=30,\\ b&=26,\\ \therefore 1+\frac{m-b}{mb}&=\frac{196}{195} \end{aligned}\right\}\begin{aligned} \text{Log.}P&=\text{log.}A-n\times\text{log.}\left(1+\frac{m-b}{mb}\right)\\ &=\text{log.}27000000-92\times\text{log.}\frac{196}{195}\\ &=7.2269858\\ &=\text{log. }16864396\text{, nearly;}\\ \therefore\ P&=16864396. \end{aligned}$$

EXAM. 3.

Suppose the population of North America to have been five millions, in the year 1800; In how many years will it amount to 16 millions, taking the *ratio of mortality* at $\frac{1}{45}$th, and the annual proportion of *births* at $\frac{1}{24}$th?

$$\left.\begin{aligned} \text{Here } A&=16000000,\\ P&=\ 5000000,\\ m&=45,\\ b&=24;\\ \therefore 1+\frac{m-b}{mb}&=\frac{367}{360}. \end{aligned}\right\}\begin{aligned} n&=\frac{\text{log.}A-\text{log. }P}{\text{log. }\left(1+\dfrac{m-b}{mb}\right)}\\ &=\frac{\text{log. }16000000-\text{log.}5000000}{\text{log.}\dfrac{367}{360}}\\ &=\frac{.5051500}{.0083636}=60.3\text{ years.} \end{aligned}$$

EXAM. 4.

The population of a province in the year 1760, was estimated at 500000 persons; in the year 1800, it amounted to 720000; from the bills of mortality it appeared, that, upon an average, $\frac{1}{50}$th part of the population had *died* annually; no register had been kept of the *births*; What was the annual proportion of *them* during this period?

$$\left.\begin{aligned} &\text{Here}\\ &A=720000,\\ &P=500000,\\ &m=50,\\ &n=40. \end{aligned}\right\}\begin{aligned} \text{Log.}\left(1+\frac{m-b}{mb}\right)&=\frac{\text{log. }A-\text{log. }P}{n},\\ \text{or log.}\left(1+\frac{50-b}{50b}\right)&=\frac{\text{log.}720000-\text{log.}500000}{40}\\ &=.0039590=\text{log. }1.009. \end{aligned}$$

Hence

$$\text{Hence } 1+\frac{50-b}{50b}=1.009=1+\frac{9}{1000},$$

$$\text{and } \frac{50-b}{50b}=\frac{9}{1000};$$

$$\therefore\ 50000-1000b=450b,$$

$$\text{or } b=\frac{50000}{1450}=34.4.$$

The annual proportion of *births*, therefore, was about $\frac{1}{34}$th.

192. But "in any country, under *given* circumstances of "births and mortality, the fraction $\frac{m-b}{mb}$ is always a *given* quantity; Let it be represented by $\frac{1}{p}$; then the relation between "the four quantities A, P, p, n, is expressed by $A=P\left(1+\frac{1}{p}\right)^n$. "If $A=mP$, we have $mP=P\left(1+\frac{1}{p}\right)^n$, or $m=\left(1+\frac{1}{p}\right)^n$; "and *taking the logarithm*, $\log. m=n\times\log.\left(1+\frac{1}{p}\right)$; from "which we deduce the *six* following formulæ."

I. $\text{Log. } A=\log. P+n\log.\left(1+\frac{1}{p}\right)$.

II. $\text{Log. } P=\log. A-n\log.\left(1+\frac{1}{p}\right)$.

III. $n=\dfrac{\log. A-\log. P}{\log.\left(1+\frac{1}{p}\right)}$.

IV. $\text{Log. }\left(1+\frac{1}{p}\right)=\dfrac{\log. A-\log. P}{n}$.

V. $n=\dfrac{\log. m}{\log.\left(1+\frac{1}{p}\right)}$, for finding the *period* in which the population would be increased *m* times.

VI. $\text{Log. }\left(1+\frac{1}{p}\right)=\dfrac{\log. m}{n}$, for finding the rate $\left(\frac{1}{p}\right)$ at which the population would be increased *m times* in *n years*.

The following Questions are intended to illustrate the use of these formulæ, in the order in which they stand.

QUESTION 1.

Suppose the population of a country to begin with *six* persons, and to increase annually by $\frac{1}{16}$th of the whole; What will be the state of its population, at the end of 200 years.

ANSWER, 1106400 persons.

QUESTION 2.

If (as stated in the 3d Example) the population of North America was five millions in the year 1800, and the rate of increase had been $\frac{7}{360}$th for 50 years previous; What was the state of its population in the year 1750?

ANSWER, 1908930 persons.

QUESTION 3.

Suppose the population of an empire to be 40 millions, and the annual increase $\frac{1}{195}$th; How long will it be before it amounts to 50 millions?

ANSWER, 43.6 years.

QUESTION 4.

What must be the *rate of increase*, that the population of a country may be changed from 1106400 persons to five millions, in 100 years?

ANSWER, about $\frac{1}{66}$th, annually.

QUESTION 5.

By means of the formula $n = \dfrac{\log. m}{\log.\left(1+\frac{1}{p}\right)}$; *verify* the following Table.

$\frac{1}{p}$	Period of *doubling*	Period of *trebling*	Period of being increased 10 *times.*
$\frac{1}{120}$	$83\frac{1}{2}$ years	$132\frac{1}{3}$ years	277.4 years
$\frac{1}{52}$	$36\frac{1}{2}$ years	57.6 years.	120.8 years

QUESTION 6.

What must be the annual increase of population in any country, that it may *double* itself every century?

ANSW. Between $\frac{1}{143}$d and $\frac{1}{144}$th.

193. M. Humboldt affirms, that in 1793 the population of New Spain amounted to 4483529 persons; and having found, from authenticated registers of births and deaths, that the *former* are about $\frac{1}{17}$th, and the *latter* about $\frac{1}{30}$th of the whole population, he estimates the population in 1803 at 5800000 persons; and afterwards asserts that in 1808 it may be expected to amount to *six millions and a half*. In this case, $P=4483529$, $m=30$, $b=17$, and $1+\frac{m-b}{mb}=\frac{523}{510}$. If $n=10$, log. $A=$log. 4483529 + 10 (log. 523 − log. 510) $=6.7609349$ = log. 5766800; and if $n=15$, then log. $A=6.8155924$ = log. 6540220. M. Humboldt's statement therefore accords very nearly with this method of calculation; as it appears from it, that in 1803 the population would amount to 5766800, and in 1808 to 6540220 persons.

LIV.

On the Solution of Exponential Equations.

194. *Exponential* equations are such as contain quantities with *unknown* or *variable* indices; and their solution consists in making such a separation of the known and unknown quantities, as to express the latter in terms of the former. The method of doing this will be understood by attending to the solution of the following Examples.

EXAMPLE 1.

Find the value of x in the equation $a^x=b$.

Taking the *logarithm* of the equation $a^x=b$, we have $x.\log. a=\log. b$, $\therefore x=\frac{\log. b}{\log. a}$; thus, let $a=5$, $b=100$, then in the equation $5^x=100$, $x=\frac{\log. 100}{\log. 5}=\frac{2.0000000}{0.6989700}=2.864$.

EXAM. 2.

To find the value of x in the equation $a^{b^x}=c$.

Assume[a] $b^x=y$, then $a^y=c$, and $y.\log. a=\log. c$, $\therefore y=\frac{\log. c}{\log. a}$; hence $b^x=\frac{\log. c}{\log. a}$ (which let) $=d$. Take the logarithm of the equation $b^x=d$, then (by Ex. 1.) $x=\frac{\log. d}{\log. b}$; thus, let $a=9$, $b=3$, $c=1000$, then in the equation $9^{3^x}=1000$, $\frac{\log. c}{\log. a}=\frac{\log. 1000}{}=3.14(=d)$; and $x=\frac{\log. d}{\log. b}=\frac{\log. 3.14}{\log. 3}=\frac{.4969296}{.4771213}=1.04$.

EXAM. 3.

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[a] In considering the nature of an exponential of the form a^{b^x}, it should be recollected that it means, *a to the power of* b^x, and not a^b *to the power of x*.

EXAM. 3.

Find the value of x in the equation $\frac{ab^x+c}{d}=e$.

By *reduction* $ab^x=de-c$, $\therefore$ log. $ab^x=$ log. $(de-c)$; and by Arts. 173, 175, log. $a+x$.log. $b=$ log. $(de-c)$; hence x. log. $b=$ log. $(de-c)-$ log. a, and $x=\frac{\text{log. }(de-c)-\text{log. }a}{\text{log. }b}$.

EXAM. 4.

Find the value of x in the equation $a_x=\frac{\sqrt{b^2-c^2}}{\sqrt[4]{d^3e}}$.

Taking the logarithm, we have $x\times$log. $a=$ log. $\sqrt{b^2-c^2}$ $-$log. $(d^3e)^{\frac{1}{4}}=\frac{1}{2}$log. $(b^2-c^2)-\frac{1}{4}$log. $d^3e=\frac{1}{2}$ log. $(b+c)(b-c)$ $-\frac{1}{4}(3$ log. $d+$ log. $e)=\frac{1}{2}$ log. $(b+c)+\frac{1}{2}$ (log. $b-c)-\frac{3}{4}$ log. d $-\frac{1}{4}$log. e; hence $x=$

$$\frac{\frac{1}{2}\text{log. }(b+c)+\frac{1}{2}\text{log. }(b-c)-\frac{3}{4}\text{log. }d-\frac{1}{4}\text{log. }e}{\text{log. }a}.$$

EXAM. 5.

Make such a separation of the quantities in the equation $(a^2-b^2)^x=a+b$, *as to shew that* $\frac{x}{1-x}=\frac{\text{log. }(a+b)}{\text{log. }(a-b)}$.

Taking the logarithm, we have

$$x.\text{log. }(a^2-b^2)=\text{log. }(a+b),$$
$$\text{or } x.\text{log. }(a+b)(a-b)=\text{log. }(a+b),$$
$$\text{i.e. } x.\text{log. }(a+b)+x.\text{log. }(a-b)=\text{log. }(a+b).$$
$$\text{Hence } x.\text{log. }(a-b)=\text{log. }(a+b)-x.\text{log. }(a+b)=$$
$$[(1-x)\text{ log. }(a+b);$$
$$\therefore \frac{x}{1-x}=\frac{\text{log. }(a+b)}{\text{log. }(a-b)}.$$

EXAM. 6.

Find the value of x in the equation $\frac{1}{5}a^x+\frac{1}{3}=\frac{2}{15}a^x+1$.

By transposition, $\left(\frac{1}{5}-\frac{2}{15}\right)a^x=1-\frac{1}{3}$, or $\frac{1}{15}a^x=\frac{2}{3}$; $\therefore a^x=10$, and x.log. $a=$ log. $10=1$; hence $x=\frac{1}{\text{log. }a}$.

EXAM. 7.

EXAM. 7.

$$\left.\begin{array}{r}\text{Given } \log. x+\log. y=\frac{5}{2},\\ \text{and } \log. x-\log. y=\frac{1}{2},\end{array}\right\}\text{to find the values of } x \text{ and } y.$$

By *addition*, $2\log. x=\frac{5}{2}+\frac{1}{2}=3$, but $2.\log. x=\log. x^2$; therefore $\log. x^2=3$, and (by Art. 181) $x^2=1000$[a]; $\therefore x=\sqrt{1000}=31.62$. By *subtraction*, $2.\log. y=\frac{5}{2}-\frac{1}{2}=2$; $\therefore \log. y^2=2$, and (by Art. 181) $y^2=100$; $\therefore y=10$.

LV.

195. We have thus endeavoured to explain, in as concise a manner as the subject will admit of, the nature and use of logarithms, with the method of constructing logarithmic tables; and have shewn their application to a few common arithmetical purposes. But it is not in its application to such purposes as these, that we can form a just and adequate idea of the value and importance of the logarithmic calculus; its utility discovers itself in every branch of mathematical science; and in point of extensiveness, there is no invention of modern times at all to be compared with it. The great purpose to which it was first applied, was to facilitate and expedite trigonometrical calculations; but it has since been had recourse to, with eminent success, in the most abstruse and intricate analytical investigations; and in this respect the subject is well worthy the attention of every Mathematical Student, as one of the best means of laying a solid foundation for his future studies.

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[a] For x^2 is the number whose *logarithm* is 3, i.e. 1000.

THE END.

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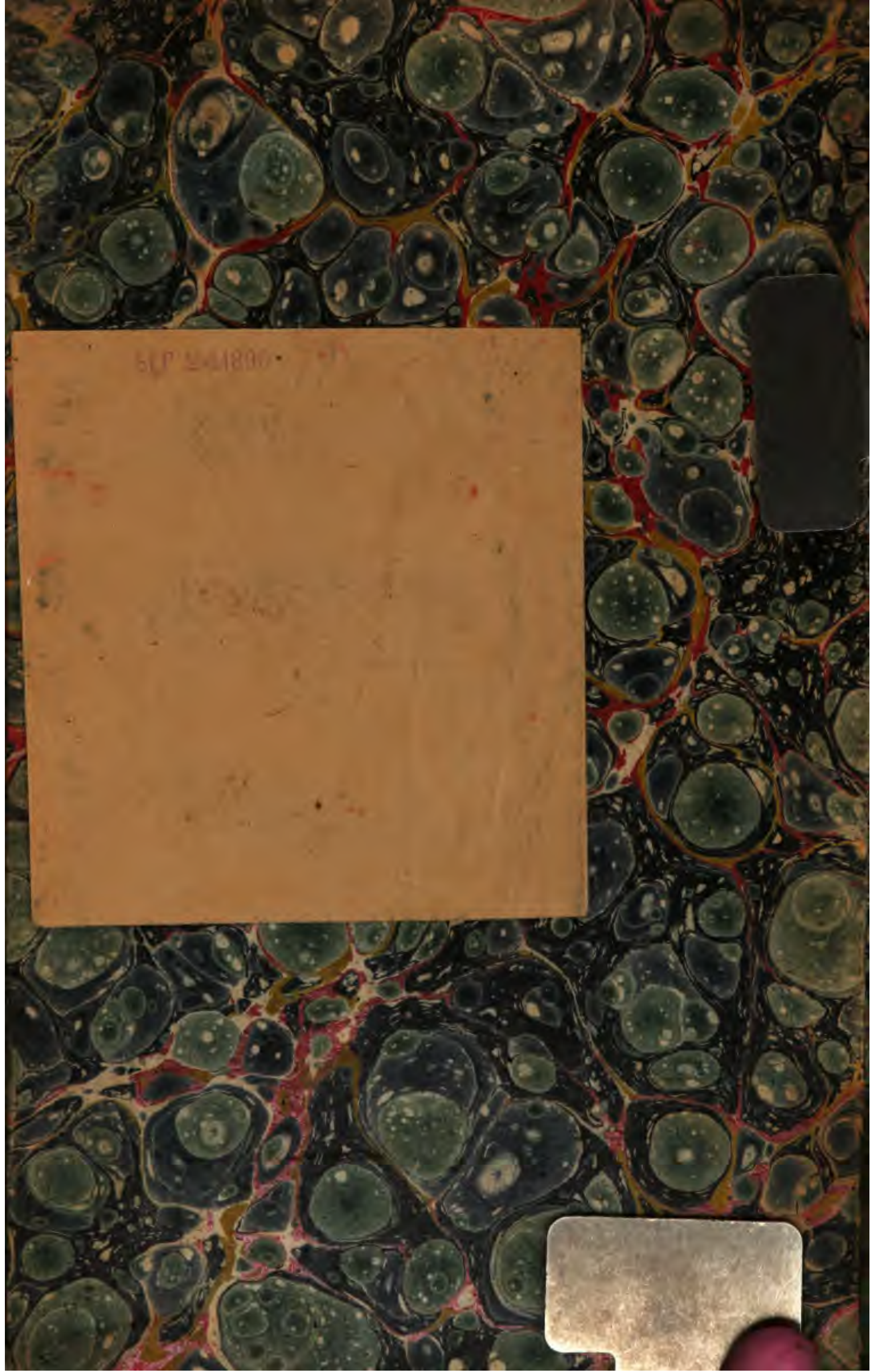

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